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The following work has been completed: The recovery of irregularly samples band-limited functions via tempered splines. Band limited functions can be recovered from their values on certain irregularly distributed discrete sampling sets as the limits of the piecewise polynomial spline interpolants when the order of the splines goes to infinity. This significant extension of the classical case when the sampling set is a lattice which was considered by L. Collatz, W. Quade, I.J. Schoenberg, and others. Orthogonality criteria for compactly supported scaling functions. The question of whether the integer translates of the scaling function constructed from a prescribed scaling sequency in the standard way are mutually orthogonal is quite subtle. The various conditions and the supporting arguments which are currently in the literature are very complicated.

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FINAL TECHNICAL REPORT

October, 1994

Multivariate Wavelet Representations and Approximations

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Report Summary

This final report summarizes activities and work completed between May, 1990 and May, 1994.

During this period we have participated in eleven scientific conferences and advanced workshops concerning various aspects of wavelets and signal processing. At each of these meetings we have presented the results of our investigations under this grant. As another output of these investigations we have produced 23 original scientific articles on various aspects of wavelets and their applications. For the record we list these articles.

1. Multiresolution analysis, Haar Bases, and Self-Similar Tilings of R^n,
   IEEE Trans. Info. Theory, 38 (1992), 556-568.
   Grochenig and Madych

2. Translation Invariant Multiresolution Analyses,
   Recent Advances in Fourier Analysis and Its Applications, Byrnes and Byrnes, eds., NATO ASI Series C, Vol. 315, Kluwer, Dordrecht, 1990, 455-462.
   Madych

3. Wilson Bases and Modulation Spaces,
   Math. Nachr., 155 (1992), 7-17.
   Feichtinger, Grochenig, and Walnut

4. Non-Orthogonal Wavelet and Gabor Expansions and Group Representations,
   Wavelets and Applications, R. Coifman et al., eds., Jones and Bartlett, Boston, 1992, 359-398.
   Feichtinger and Grochenig

5. Image Reconstruction in Hilbert Space,
   Mathematical Methods in Tomography, G.T. Herman, A.K. Louis, F. Natterer, eds., Lecture Notes in Mathematics 1497, Springer Verlag, Berlin, 1991, 15-49.
   Madych

6. Wavelets and Generalized Box Splines,
   Appl. Analysis, 44 (1992), 51-76.
   Lorentz and Madych

7. Spline Wavelets for Ordinary Differential Equations,
   GMD technical report No. 562.
   Lorentz and Madych

8. Multiresolution analyses, Tiles, and Scaling Functions,
   Probabilistic and Stochastic Methods in Analysis with Applications, Byrnes *et al.*, eds., NATO ASI Series C, Vol. 375, Kluwer, Dordrecht, 1992, 233-243.
   Madych

9. A Riesz Basis for Bargman-Fock Space Related to Sampling and Interpolation,
   Arkiv. f. Math., 30 (1992), 283-295.
   Grochenig and Walnut

10. Reconstruction Algorithms in Irregular Sampling,
    Math. Comp., 59, (1992), 181-194.
    Grochenig

11. Efficient Algorithms in Irregular Sampling of Band-Limited Functions.
    Proc. Int. Phoenix Conf. on Computers and Communication 1991, IEEE Comp. Soc, 1991, 490-495.
    Grochenig

12. Some elementary properties of multiresolution analyses of $L^2(R^n)$,
    Wavelets - A tutorial in Theory and Applications, C. K. Chui, ed., Academic Press, Boston, 1992, 259-294.
    Madych

13. Self-similar Lattice Tilings,
    J. Fourier Analysis Appl. to appear.
    Haas and Grochenig

14. Miscellaneous Error Bounds for Multiquadric and Related Interpolators,
    Computers Math. Applic., Vol. 24, No.12 (1992), 121-138.
    Madych

15. Gabor Wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view,
    Wavelets - A tutorial in Theory and Applications, C. K. Chui, ed., Academic Press, Boston, 1992, 359-298.
    Feichtinger and Grochenig

16. Sharp results on random sampling of band-limited functions,
Probabilistic and Stochastic Methods in Analysis with Applications, Byrnes *et al.*, eds., NATO ASI Series C, Vol. 375, Kluwer, Dordrecht, 1992, 323-335.
Grochenig

17. A discrete theory of irregular sampling,
Lin. Alg. Appl., 193 (1993), 129-150.
Grochenig

18. Irregular sampling of wavelet and short time Fourier transforms,
Constr. Approx., 9 (1993), 283-297.
Grochenig

19. Acceleration of the Frame Algorithm,
IEEE Trans, Signal Proc., 41 (1993), 3331-3340.
Grochenig

20. The recovery of irregularly sampled band-limited functions via tempered splines,
J. Funct. Anal. 125 (1994), 201-222.
Lyubarskii and Madych

21. Orthogonality criteria for compactly supported scaling functions,
Appl. Comp. Harm. Anal., 1 (1994), 242-245.
Grochenig

22. Orthogonal Wavelet Bases for $L^2(R^n)$,
Fourier Analysis: Analytic and Geometric Aspects, Bray *et al.*, eds., Marcel Dekker, New York, 1994, 243-302.
Madych

23. Scaling Functions and Sequences Associated with Orthonormal Wavelets,
Houston J. Math. to appear.
Dlin and Madych

We bring attention to the fact that twenty of these articles have already appeared in various scholarly journals and books. The first twenty one articles on this list together with detailed summaries have been already submitted with earlier technical reports. The final two article on this list are appended at the end of this report and are summarized below.

- *Orthogonal wavelet bases for* $L^2(R^n)$. An orthogonal wavelet basis consists of dilates and translates of one function or, more generally, a relatively small finite number of functions. In this presentation we give the details of the construction of such bases where the dilation is defined via a fairly general linear transformation. A liberal number of examples are given to illustrate the flexibility of the recipe. The significance of this work lies in the fact that, unlike the dyadic case, in the case of a more general dilation matrix there is no prescription for constructing wavelets from a given scaling function or sequence. In this article we introduce several methods for constructing wavelets with desired properties in the case of general dilation matrices.

- *Scaling Functions and Sequences Associated with Orthonormal Wavelets.* It is well known that the integer translates of the scaling function associated with a given scaling sequence may fail to be mutually orthogonal. In this article, we address several technical questions related to this phenomenon. For instance, we show that the scaling function naturally associated with a finite scaling sequence always generates a multiresolution analysis and give elementary but non-trivial examples of scaling sequences which give rise to pathological scaling functions. The significance of this work lies in the fact that it is very important in various applications to know whether an apparent scaling sequence will generate orthogonal wavelets via the usual paradigm.

In the previous technical report we mentioned that we were preparing a paper on results concerning the breakdown of the so-called scaling functions into more elementary building blocks. Since that report we have discovered that these building blocks are closely related to the class of distributions which are invisible or undetectable by the corresponding family of wavelets. (Namely, if f is such a distribution and $\{\psi_{j,k}\}_{j,k\in Z}$ is the corresponding family of orthogonal wavelets then the scalar products $\langle f, \psi_{j,k}\rangle$ are zero for all j and k.) We believe that these observations can be developed into a significant theory providing a deeper understanding of wavelets and the multiresolution analysis paradigm. The principal investigator intents to continue his investigations into this area.

Orthogonal wavelet bases for $L^2(R^n)$

W. R. Madych*

Abstract

An orthogonal wavelet basis consists of dilates and translates of one function or, more generally, a relatively small finite number of functions. In this presentation we give the details of the construction of such bases where the dilation is defined via a fairly general linear transformation. A liberal number of examples are given to illustrate the flexibility of the recipe.

1 Introduction

1.1 Overview

In this talk we present the notion of orthogonal wavelet bases of $L^2(\mathbb{R}^n)$ relative to a general dilation matrix A and outline the natural scheme for constructing such bases. This scheme was originally developed by S. Mallat and Y. Meyer, see [31, 33].

An orthogonal wavelet basis for $L^2(\mathbb{R})$ relative to dyadic dilation is a complete orthonormal system of the form

(1) $$\{2^{k/2}\psi(2^k x - j)\}_{k,j}$$

where ψ is an appropriate function in $L^2(\mathbb{R})$ and the indicies k and j run through the integer lattice $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. The elements of such a basis are often referred to as wavelets and the function ψ is sometimes referred to as the *fundamental wavelet.*

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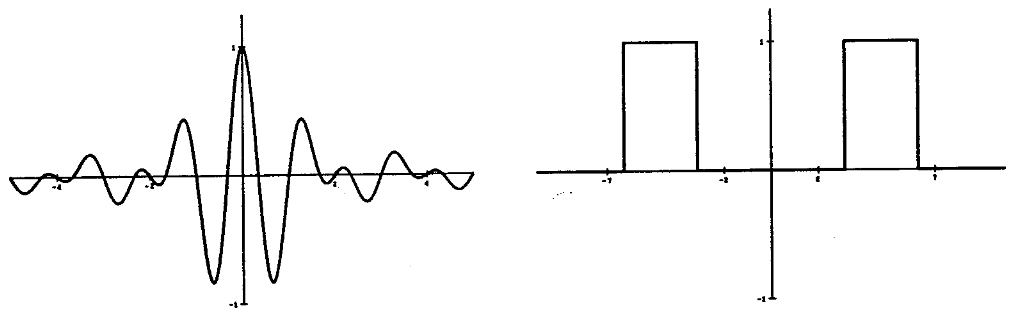

Figure 1: The sinc wavelet and its Fourier transform

1.2 Examples

1.2.1

Consider the classical example

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding system (1) is the well known Haar basis for $L^2(\mathbb{R})$.

1.2.2

Another example is the case

$$\psi(x) = \frac{\sin 2\pi x - \sin \pi x}{\pi x}$$

which is related to the theory of band limited functions and cardinal series. Since the Fourier transform of ψ is the characteristic function of the set $\{\xi : \pi \le |\xi| \le 2\pi\}$ it follows from routine functional analytic and Fourier transform techniques that the corresponding set (1) is a complete orthonormal system for $L^2(\mathbb{R})$. See Figure 1 .

Because of the close relationship of ψ to the classical cardinal sine, also known as the sinc or *sinus cardinalis*,

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x} \, ,$$

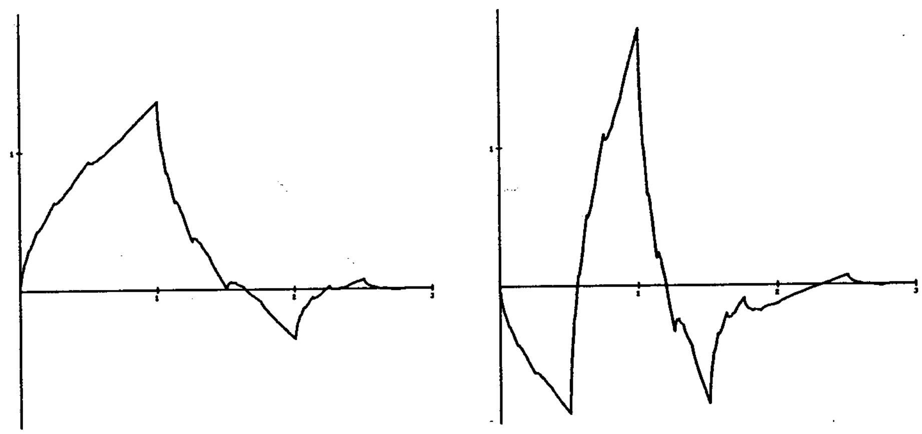

Figure 2: The function ϕ and the wavelet ψ of Example 1.2.3

we refer to this basis as the sinc wavelet basis.

1.2.3

Finally consider the function whose graph is the second plot given in Figure 2. This function is defined by

$$\psi(x) = s_3\phi(x) - s_2\phi(x-1) + s_1\phi(x-2) - s_0\phi(x-3)$$

where ϕ is the unique compactly supported solution of

$$\phi(x) = s_0\phi(x) + s_1\phi(x-1) + s_2\phi(x-2) + s_3\phi(x-3)$$

with $L^2(I\!R)$ norm one, see the first plot in Figure 2, and

$$\{s_0, s_1, s_2, s_3\} = \left\{\frac{1+\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right\} .$$

The function ψ is a member of a class of fundamental wavelets introduced by Ingrid Daubechies [12].

1.3 Dilation matrices and wavelets

In the description (1) note the role played by the translations $x \to x + j$, $j \in \mathbb{Z}$ and the dilation $x \to 2x$ which is expansive and maps the lattice $\mathbb{Z}$ into itself. Thus, more generally, if Γ is a lattice in $\mathbb{R}^n$, and A is an expansive linear transformation on $\mathbb{R}^n$ which leaves Γ invariant then a *wavelet basis associated to* (Γ, A) is a complete orthonormal system for $L^2(\mathbb{R}^n)$ of the form

$$\{a^{k/2}\psi_j(A^k x - \gamma)\}_{j,k,\gamma} \tag{2}$$

where $\psi_1, \ldots, \psi_m$ are an appropriate collection of orthonormal functions in $L^2(\mathbb{R}^n)$, $a = |\det A|$, the index k runs through the integers $\mathbb{Z}$, and the parameter γ runs through the lattice Γ. Wavelets are simply the members of such a basis and the collection $\{\psi_1, \ldots, \psi_m\}$ is the set of fundamental wavelets.

1.4 Examples

1.4.1

In the case $n = 1$, $\Gamma = \mathbb{Z}$, and $A = 3$ consider the functions ψ_l, $l = 1, 2$, defined by

$$\psi_l(x) = \begin{cases} u_{lk} & \text{if } k/3 \le x < (k+1)/3, \quad k = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

where the u_{lk}'s are scalars chosen so that

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{pmatrix}$$

is an orthogonal matrix. This definition implies that the system of functions

$$\{3^{k/2}\psi_l(3^k x - j)\}_{l \in \{1,2\}, k \in \mathbb{Z}, j \in \mathbb{Z}}$$

is an orthonormal system in $L^2(\mathbb{R})$; that it is complete follows from an argument similar to that used to show the completeness of the Haar system. Here $l = 1, 2$ and both k and j run through $\mathbb{Z}$.

More generally if N is an integer greater than one and

$$\frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_{1,0} & u_{1,1} & \cdots & u_{1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N-1,0} & u_{N-1,1} & \cdots & u_{N-1,N-1} \end{pmatrix}$$

is an orthogonal matrix then the collection of functions $\{\psi_1, \ldots, \psi_{N-1}\}$ defined by

$$\psi_l(x) = \begin{cases} u_{l,k} & \text{if } k/N \leq x < (k+1)/N, \quad k = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

for $l = 1, \ldots, N-1$ is a set of fundamental wavelets. The corresponding wavelet basis of $L^2(\mathbb{R}^n)$ is the collection

$$\{N^{k/2}\psi_l(N^k x - j)\}_{l,k,j}$$

where $l = 1, \ldots, N-1$ and both k and j run through $\mathbb{Z}$.

1.4.2

In the case $n = 2$, $\Gamma = \mathbb{Z}^2$, and

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

consider the function

$$\psi(x) = 2\,\mathrm{sinc}(Ax) - \mathrm{sinc}(x)$$

Where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathrm{sinc}(x) = \mathrm{sinc}(x_1)\,\mathrm{sinc}(x_2) \ .$$

The function ψ is a bi-variate analog of the fundamental wavelet in Example 1.2.2. The Fourier transform of ψ is the characteristic function of the region $\{\xi \ : \ \max\{|\xi_1|, |\xi_2|\} \geq \pi, \ \ |\xi_1| + |\xi_2| \leq 2\pi\}$, see Figure 3. That

$$\{2^{k/2}\psi(A^k x - j)\}_{k \in \mathbb{Z}, j \in \mathbb{Z}^2}$$

is a wavelet basis of $L^2(\mathbb{R}^2)$ follows from reasoning identical to that used in Example 1.2.2.

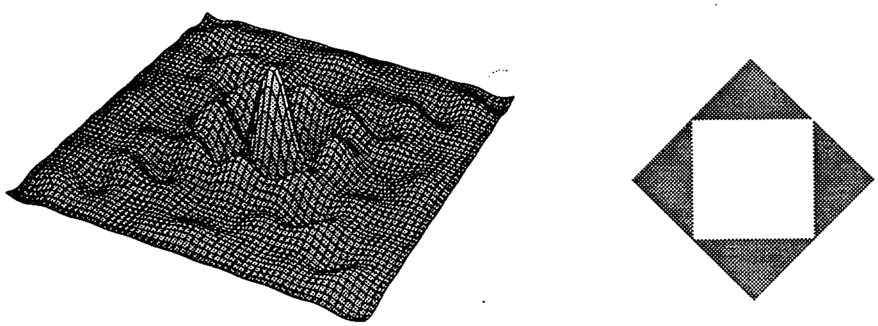

Figure 3: The wavelet ψ and the support of $\hat{\psi}$ of Example 1.4.2

1.5 Contents and Notation

As mentioned in the overview, a natural framework for the construction of a wavelet basis called a multiresolution analysis was developed by Y. Meyer and his collaborators, see [20, 24, 31, 32, 33]. In view of form of these bases, (2), it should not be surprising that the main elements of such analyses are the notions of dilation and translation relative to a lattice.

In Section 2 we give an outline of the theory and include several examples of mutiresolution analyses of $L^2(\mathbb{R}^n)$ associated with a lattice Γ and an appropriate dilation matrix A. In Section 3 we show how these analyses give rise to orthogonal wavelet bases. Miscellaneous remarks and acknowledgements are collected in Section 4.

We use standard mathematical terminology and notation. A brief list of some of the conventions used here follows: The Fourier transform $\hat{f}$ of an integrable function f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle \xi, x\rangle} f(x)dx$$

and distributionally otherwise. Basic facts concerning Fourier transforms and distributions will be used without further elaboration in what follows. To avoid the pedantic repetition of "almost everywhere" and other modifying phrases which are inevitably necessary when dealing with functions defined almost everywhere, all equalities between functions and other related notions are interpreted in the distributional sense whenever possible. The term support is also used in the distributional sense; in particular the support of a function f in $L^2(\mathbb{R}^n)$ is a well defined closed set. If W is a collection of tempered distributions then $\widehat{W}$ is the collection of Fourier transforms of elements of W, in other words $\widehat{W} = \{f : f = \hat{g} \text{ for some } g \text{ in } W\}$. For a subset Ω of $\mathbb{R}^n$, a linear transformation B on $\mathbb{R}^n$, and an element y of $\mathbb{R}^n$ the sets $B\Omega$ and $\Omega + y$ are defined by $B\Omega = \{x : x = B\omega \text{ for some } \omega \text{ in } \Omega\}$ and $\Omega + y = \{x : x = \omega + y \text{ for some } \omega \text{ in } \Omega\}$; $L^2(\Omega)$ is the L^2 closure of the subspace of those functions in $L^2(\mathbb{R}^n)$ whose support is contained in Ω. Given a measurable set Ω, χ_Ω denotes its characteristic or indicator function and $|\Omega|$ denotes its Lebesgue measure.

2 Multiresolution Analyses

2.1 Definitions

Suppose Γ is a lattice in $\mathbb{R}^n$, that is, Γ is the image of the integer lattice $\mathbb{Z}^n$ under some nonsingular linear transformation. We say that a linear transformation A on $\mathbb{R}^n$ is an *acceptable dilation for* Γ if it satisfies the following properties:

- A leaves Γ invariant. In other words, $A\Gamma \subset \Gamma$ where $A\Gamma = \{y : y = Ax \text{ and } x \in \Gamma\}$.
- All the eigenvalues, λ_i, of A satisfy $|\lambda_i| > 1$.

These properties imply that $|\det A|$ is an integer a which is ≥ 2. For example, if $A = pI$ where p is an integer ≥ 2 and I is the identity then A is an acceptable dilation for any lattice Γ and $a = |\det A| = p^n$.

Such an A induces a unitary dilation operator $U_A : f \longrightarrow U_A f$ on $L^2(\mathbb{R}^n)$, defined by

$$U_A f(x) = |\det A|^{-1/2} f(A^{-1}x) . \tag{3}$$

If V is a subspace of $L^2(\mathbb{R}^n)$ we use the customary notation $U_A V$ to denote the image of V under U_A, that is, $U_A V = \{f : f = U_A g,\ g \in V\}$. The translation operator τ_y is defined by $\tau_y f(x) = f(x-y)$.

A *multiresolution analysis* $\mathcal{V}$ associated with (Γ, A) is a family $\{V_j\}_{j\in Z}$ of closed subspaces of $L^2(\mathbb{R}^n)$ which enjoys the following properties:

A1. $V_j \subset V_{j+1}$ for all j in Z.

A2. $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$.

A3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

A4. $f(x) \in V_j$ if and only if $f(Ax) \in V_{j+1}$. In other words

$$V_j = U_A^{-j} V_0,\ j \in \mathbb{Z}.$$

A5. V_0 is invariant under τ_γ. More specifically, if $f(x)$ is in V_0 then so is $f(x-\gamma)$ for all γ in Γ.

A6. There is a function $\phi \in V_0$, called the *scaling function*, such that $\{\tau_\gamma \phi\}_{\gamma\in\Gamma}$ is a complete orthonormal basis for V_0.

The case $A = 2I$ is often referred to as a *dyadic* multiresolution analysis. It is also the case to which most of the current literature is devoted and, except for certain technicalities, is representative of the general case.

A multiscale analysis is said to be *composed of generalized spline functions* in the sense of Meyer if all the elements of the subspace V_0 are continuous and the mapping which maps f into the sequence of values $\{f(\gamma)\}_{\gamma\in\Gamma}$ is an isomorphism from V_0 onto $l^2(\Gamma)$.

Remark The above definition has several redundancies and thus is not very compact, for example see [30]. However we use it here for reasons of tradition and convenience.

2.2 Lattices, cosets, and related items

Every lattice Γ in $\mathbb{R}^n$ is the image of $\mathbb{Z}^n$ under an invertible linear transformation. Thus there is no loss of generality by restricting attention to the case $\Gamma = \mathbb{Z}^n$. We do this in what follows to avoid unnecessary obfuscation.

The reader who is interested in the general statements of the results discussed below can systematically replace $\mathbb{Z}^n$ with Γ and $2\pi\mathbb{Z}^n$ with the corresponding dual lattice Γ'.

Recall that if A is an acceptable dilation of $\mathbb{Z}^n$ then $A\mathbb{Z}^n$ is a subgroup of $\mathbb{Z}^n$ and a coset of $A\mathbb{Z}^n$ is a set of the form

$$k + A\mathbb{Z}^n = \{k + Aj : j \in \mathbb{Z}^n\}$$

where k is an element of $\mathbb{Z}^n$. Any pair of cosets are either identical or disjoint so that the collection of all cosets, which is denoted by $\mathbb{Z}^n/A\mathbb{Z}^n$, consists of disjoint sets whose union is $\mathbb{Z}^n$. The number of disjoint cosets in $\mathbb{Z}^n/A\mathbb{Z}^n$ is equal to $a = |\det A|$. A subset $\mathcal{K}$ of $\mathbb{Z}^n$ is said to be a full collection of representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$ if it contains exactly a elements and

$$\bigcup_{\kappa\in\mathcal{K}} (\kappa + A\mathbb{Z}^n) = \mathbb{Z}^n.$$

We identify the representative with the coset it represents. For instance, $\kappa = 0$ refers to the element in $A\mathbb{Z}^n$ chosen to represent this coset.

Since the Fourier transform of $f(A^{-1}x)$ is $a\hat{f}(A^*\xi)$ and we will use this fact and its variants often in what follows, we adopt the notation $B = A^*$ to avoid cumbersome expressions such as $(A^*)^{-1}$.

Furthermore we write $\mathcal{K}_A$ to denote a full collection of representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$,

$$\mathcal{K}_A = \{\kappa_1, \ldots, \kappa_a\} \ .$$

Since B is an acceptable dilation for $\mathbb{Z}^n$ whenever A is, we write $\mathcal{K}_B$ to denote a full collection of representatives of $\mathbb{Z}^n/B\mathbb{Z}^n$,

$$\mathcal{K}_B = \{\nu_1, \ldots, \nu_a\} \ .$$

Observe that $\mathcal{K}_A$ need not necessarily be a full collection of representatives of $\mathbb{Z}^n/B\mathbb{Z}^n$ and vice versa. This is easily seen by considering the two dimensional example

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} .$$

Finally we note that any absolutely convergent sum over $\mathbb{Z}^n$ can be iterated as follows:

$$\sum_{j\in\mathbb{Z}^n} s_j = \sum_{\kappa\in\mathcal{K}_B} \left\{ \sum_{j\in\kappa+B\mathbb{Z}^n} s_j \right\} . \tag{4}$$

2.3 Scaling functions

An application of Plancherel's formula implies that the orthonormality of the collection $\{\phi(x-j)\}_{j\in\mathbf{Z}^n}$ is equivalent to

$$\sum_{k\in\mathbf{Z}^n} |\hat{\phi}(\xi - 2\pi k)|^2 = 1. \tag{5}$$

Thus every scaling function ϕ must satisfy (5).

The existence of a scaling function is equivalent to the following apparently weaker condition:

A6_1. There is a function ϕ in V_0 such that $\{\phi(x-k)\}_{k\in\mathbf{Z}^n}$ is a Riesz basis for V_0.

In many specific examples this property is much more apparent and easier to verify than A6. We refer to the function ϕ of condition $A6_1$ as a *pseudo-scaling function.*

We remind the reader that $\{\phi(x-k)\}_{k\in\mathbf{Z}^n}$ is a Riesz basis for its closed linear span if and only if there are positive constants, c_1 and c_2 which are independent of $\{a_k\}_{k\in\mathbf{Z}^n}$, such that

$$c_1 \sum_{k\in\mathbf{Z}^n} |a_k|^2 \leq \int_{\mathbf{R}^n} |\sum_{k\in\mathbf{Z}^n} a_k\phi(x-k)|^2 dx \leq c_2 \sum_{k\in\mathbf{Z}^n} |a_k|^2.$$

It is well known and easy to verify that $\{\phi(x-j)\}_{j\in\mathbf{Z}^n}$ is a Riesz basis for the closure of its linear span if and only if there are positive constants, c_1 and c_2, such that

$$c_1 \leq \sum_{j\in\mathbf{Z}^n} |\hat{\phi}(\xi - 2\pi j)|^2 \leq c_2. \tag{6}$$

Orthonormality of $\{\phi(x-j)\}_{j\in\mathbf{Z}^n}$ is equivalent to the case $c_1 = c_2 = 1$.

Thus the alternate item A6_1, in the definition of multiresolution analyses is equivalent to saying that V_0 is the closed linear span of the $\mathbb{Z}^n$ translates of one function ϕ which satisfies (6). The original item A6 means that the constants c_1 and c_2 satisfy $c_1 = c_2 = 1$.

2.3.1 Details

The calculational technique of periodization is very useful in the study of multiresolution analyses. It can be described as follows:

Suppose Q is a tile with respect to a lattice Γ for $I\!R^n$. That is Q is a compact subset of $I\!R^n$ which enjoys the following properties:

- $\bigcup_{\gamma\in\Gamma}(\gamma+Q) = I\!R^n$ and
- $|(\gamma+Q)\bigcap Q| = 0$ for all γ in $\Gamma\setminus\{0\}$.

Then if h is any intergrable function on $I\!R^n$ we may write

$$\int_{I\!R^n} h(\xi)d\xi = \int_Q \sum_{\gamma\in\Gamma} h(\xi+\gamma)d\xi \,. \tag{7}$$

In the applications below it will usually suffice to take Q to be the cube $Q_\pi = [-\pi,\pi]^n$ and $\Gamma = 2\pi Z\!\!Z^n$. However occassionally other variants of this formula will be used. Our first application of this elementary formula will be the proof of (5).

To see (5) note that by virtue of Plancherel's formula , (7), and the fact that the collection $\{\phi(x-j)\}_{j\in\mathbf{Z}^n}$ is orthonormal we may write

$$\int_{I\!R^n} \phi(x-j)\overline{\phi(x-k)}dx = \int_{Q_\pi} \sum_{m\in\mathbf{Z}^n} |\hat{\phi}(\xi+2\pi m)|^2 e^{-i(j-k,\xi)}d\xi = \delta_{j,k}$$

where $\delta_{j,k}$ is the Kronecker delta. The last equality implies (5).

To see that A6$_1$ implies A6 suppose $\{\phi(x-k)\}_{k\in\mathbf{Z}^n}$ is a Riesz basis for V_0. In view of (6)

$$\frac{1}{\sqrt{\sum_{j\in\mathbf{Z}^n}|\hat{\phi}(\xi-2\pi j)|^2}}$$

is a bounded $2\pi Z\!\!Z^n$ periodic function. Since f is in V_0 if and only if it enjoys the representation

$$\hat{f}(\xi) = F(\xi)\hat{\phi}(\xi)$$

for some F in $L^2(I\!R^n/2\pi Z\!\!Z^n)$ it is clear that the function ϕ_0 defined by

$$\hat{\phi}_0(\xi) = \frac{\hat{\phi}(\xi)}{\sqrt{\sum_{j\in\mathbf{Z}^n}|\hat{\phi}(\xi-2\pi j)|^2}}$$

is in V_0. Now ϕ_0 satisfies (5) so the collection $\{\phi_0(x-j)\}_{j\in\mathbf{Z}^n}$ is an orthonormal subset of V_0. That the collection $\{\phi_0(x-j)\}_{j\in\mathbf{Z}^n}$ is complete follows from the fact that every function f which enjoys the above representation in terms of ϕ also enjoys an analogous representaion in terms of ϕ_0.

2.3.2 A property of the scaling function related to density

Because of condition $A3$ the scaling function ϕ must satisfy

$$\lim_{k\to\infty} \frac{1}{|B^{-k}Q|} \int_{B^{-k}Q} |\hat{\phi}(\xi)|^2 d\xi = 1 \tag{8}$$

for every cube Q of finite diameter in $\mathbb{R}^n$. A detailed explanation of this may be found, for example, in [30]. Thus if $\hat{\phi}$ is continuous at the origin then

$$|\hat{\phi}(0)| = 1 \ . \tag{9}$$

In particular if ϕ is integrable over $\mathbb{R}^n$ then

$$|\int_{\mathbb{R}^n} \phi(x)dx| = |\hat{\phi}(0)| = 1 \ .$$

2.4 Scaling sequences

In view of A1 and A4 the scaling function ϕ must satisfy the two scale difference equation

$$\phi(x) = \sum_{k\in\mathbb{Z}^n} s_k \phi(Ax - k) \tag{10}$$

for some sequence $\{s_k\}_{k\in\mathbb{Z}^n}$ in $l^2(\mathbb{Z}^n)$. The sequence $\{s_k\}_{k\in\mathbb{Z}^n}$ is called the *scaling sequence.* The Fourier transform of (10) is

$$\hat{\phi}(\xi) = S(B^{-1}\xi)\hat{\phi}(B^{-1}\xi) \tag{11}$$

where $B = A^*$ is the adjoint of A and $S(\xi)$ is the $2\pi\mathbb{Z}^n$ periodic function

$$S(\xi) = \frac{1}{a} \sum_{k\in\mathbb{Z}^n} s_k e^{-i\langle k,\xi\rangle}.$$

2.4.1 Properties of scaling sequences

As a consequence of (5) the periodic function S introduce above must satisfy

$$\sum_{\kappa\in\mathcal{K}_B} |S(\xi - 2\pi B^{-1}\kappa)|^2 = 1 \tag{12}$$

where $\mathcal{K}$ is any full collection of representatives of $\mathbb{Z}^n/B\mathbb{Z}^n$. Furthermore if $\hat{\phi}(\xi)$ is continuous at the origin then $S(\xi)$ is also continuous at the origin and

$$S(0) = 1 \ . \tag{13}$$

This follows from (11) and the fact that, in view of (9), $\hat{\phi}(0) \neq 0$.

In terms of the scaling sequence (12) is equivalent to

$$\sum_{k \in \mathbb{Z}^n} s_{k-Aj}\overline{s_k} = a\delta_{j,k} \ . \tag{14}$$

If this sequence is also in $l^1(\mathbb{Z}^n)$ then (13) is equivalent to

$$\sum_{k \in \mathbb{Z}^n} s_k = a \ . \tag{15}$$

Thus (14) and (15) are a necessary set of conditions for an $l^1(\mathbb{Z}^n)$ sequence to be a scaling sequence.

Details To see (12) use (5), (11), iteration (4), the fact that $S(B^{-1}\xi)$ is $2\pi B\mathbb{Z}^n$ periodic, and (5) again to write

$$\begin{aligned} 1 &= \textstyle\sum_{j \in \mathbb{Z}^n} |S(B^{-1}(\xi - 2\pi j)\hat{\phi}(B^{-1}(\xi - 2\pi j)|^2 \\ &= \textstyle\sum_{\kappa \in \mathcal{K}_B} |S(B^{-1}(\xi - 2\pi\kappa)|^2 \left\{\sum_{j \in \kappa + B\mathbb{Z}^n} |\hat{\phi}(B^{-1}(\xi - 2\pi\kappa) - 2\pi j)|^2\right\} \\ &= \textstyle\sum_{\kappa \in \mathcal{K}_B} |S(B^{-1}(\xi - 2\pi\kappa)|^2 \end{aligned}$$

which is the desired result.

2.4.2 More properties of the scaling functions and sequences

Note that in view of (10) ϕ may be considered as a fixed point of the transformation $\phi \rightarrow \sum_{k \in \mathbb{Z}^n} s_k\phi(Ax - k)$. Unfortunately the solution of (10) is not unique since the distribution ϕ_0 defined by

$$\hat{\phi}_0(\xi) = h(\xi)\hat{\phi}(\xi)$$

is also a solution of (10) whenever h is any locally integrable function which satisfies $h(B\xi) = h(\xi)$. Such a distribution ϕ_0 may fail to be a scaling function. On the other hand if h also enjoys $|h(\xi)| = 1$ for almost all ξ then ϕ_0 is also a scaling function but not necessarily for the same multiresolution analysis. Further restrictions are needed on the scaling sequence $\{s_k\}_{k\in\mathbb{Z}^n}$ and the scaling function ϕ to guarantee a unique solution of equation (10), for example see [2, 13] for some details concerning this matter.

The scaling function ϕ is not unique since any function ϕ_1 whose Fourier transform satisfies

$$\hat{\phi}_1(\xi) = H(\xi)\hat{\phi}(\xi)$$

is also a scaling function for the same multiresolution analysis whenever H is any measurable $2\pi\mathbb{Z}^n$ periodic function which satisfies $|H(\xi)| = 1$ for almost all ξ. The scaling equation satisfied by ϕ_1 will, in general, be different from the one satisfied by ϕ.

2.5 Examples

2.5.1 Multiresolution analyses generated by self similar sets

Suppose $\mathcal{K}_A$ is a full collection of representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$. Consider the compact set Q defined by

$$Q = \{x \in \mathbb{R}^n : x = \sum_{j=1}^{\infty} A^{-j}\epsilon_j,\ \epsilon_j \in \mathcal{K}_A\}\ . \tag{16}$$

Note that Q depends both on A and the choice of 'digits', $\mathcal{K}_A$. This set satisfies many intersting properties. For our purposes it suffices to note the following:

- Q is self similar in the affine sense. That is,

$$AQ = \bigcup_{\kappa\in\mathcal{K}_A} (\kappa + Q) \tag{17}$$

  where the terms in the union are essentially mutually disjoint , that is $|(\kappa_1 + Q)\bigcap(\kappa_2 + Q)| = 0$ whenever $\kappa_1 \neq \kappa_2$.

- The characteristic function, χ_Q, of this set satisfies

$$\chi_Q(x) = \sum_{\kappa\in\mathcal{K}_A} \chi_Q(Ax - \kappa)\ . \tag{18}$$

This, of course, is equivalent to (17). Also note that the sequence $\{s_k\}_{k\in\mathbb{Z}^n}$ defined by

$$s_k = \begin{cases} 1 & \text{if } k \in \mathcal{K}_A \\ 0 & \text{otherwise} \end{cases}$$

satisfies properties (14) and (15).

- The measure of Q, $|Q|$, is an integer ≥ 1.

The characteristic function of Q, χ_Q, generates a multiresolution analysis in the following sense: If V_0 is the $L^2(\mathbb{R}^n)$ closure of the linear span of $\{\chi_Q(x-k)\}_{k\in\mathbb{Z}^n}$ then the collection of spaces $\mathcal{V} = \{V_j\}_{j\in\mathbb{Z}}$ defined by

$$\begin{aligned} V_j &= U_A^{-j} V_0 \\ &= \{f(x) : f(A^{-j}x) \in V_0\} \end{aligned}$$

is a multiresolution analysis associated to $(\mathbb{Z}^n, A)$. This is a consequence of the fact that χ_Q satisfies (18) and the fact that

$$\sum_{k\in\mathbb{Z}^n} |\hat{\chi}_Q(\xi - 2\pi k)|^2 > 0 \qquad \text{a.e.}$$

See [30] for more details.

Furthermore, if $|Q| = 1$ then χ_Q is a scaling function for this multiresolution analysis. In other words, $\{\chi_Q(x-k)\}_{k\in\mathbb{Z}^n}$ is a complete orthonormal system for V_0. For conditions which guarantee that $|Q| = 1$ see [17, 30].

On the other hand if $|Q| > 1$ then $\{\chi_Q(x-k)\}_{k\in\mathbb{Z}^n}$ fails to be a Riesz basis for V_0. Nevertheless it is not difficult to find a scaling function for $\mathcal{V}$. The function ϕ defined by the formula for its Fourier transform

$$\hat{\phi}(\xi) = \frac{\hat{\chi}_Q(\xi)}{\{\sum_{k\in\mathbb{Z}^n} |\hat{\chi}_Q(\xi - 2\pi k)|^2\}^{1/2}}$$

is one such example. In many cases it is possible, and often not difficult, to find a scaling function which is the characteristic function of an appropriate set with measure one; however, it is not clear whether this is always the case.

For more details concerning these multiresolution analyses see [17, ?, 30]. We conclude this subsection with several specific examples.

Univariate examples In the case $n = 1$ consider $A = 2$. If $\mathcal{K}_A = \{0, 1\}$ then Q is the interval $[0, 1]$ and χ_Q is a scaling function for the multiresolution analysis which it generates; note that

$$V_0 = \{f \in L^2(\mathbb{R}) : \ f \text{ is constant on the intervals } (j, j+1), \ j \in \mathbb{Z}\} .$$

If $\mathcal{K}_A = \{0, m\}$ where m is an odd integer $\neq 1$ then Q is the interval $[0, m]$ and χ_Q fails to be a scaling function for the multiresolution analysis which it generates. However it is apparent that this multiresolution analysis, namely the one generated by $\chi_{[0,m]}$, is the same as the one generated by $\chi_{[0,1]}$.

If $A = -2$ and $\mathcal{K}_A = \{0, 1\}$ then Q is the interval $[-2/3, 1/3]$. The rest of the remarks made in the case $A = 2$ are valid *mutatis mutandis* in this case.

The case $A = 3$ is more interesting. If $\mathcal{K}_A = \{0, 1, 2\}$ then Q is the interval $[0, 1]$ and χ_Q is a scaling function for the multiresolution analysis which it generates; here again V_0 is the same as above. If $\mathcal{K}_A$ is a multiple of $\{0, 1, 2\}$ then Q is the same multiple of the interval $[0, 1]$ and χ_Q generates the same multiresolution analysis as $\chi_{[0,1]}$. However if, for instance, $\mathcal{K}_A = \{0, 1, 5\}$ then Q is a disconnected set of measure one and the multiresolution analysis generated by Q is very different from the one generated by $[0, 1]$; indeed here

$$V_0 = \{f \in L^2(\mathbb{R}) : \ f \text{ is constant on the sets } j + Q, \ j \in \mathbb{Z}\} .$$

More generally one may consider the case $A = N$ where N is an integer such that $|N| > 1$. The cases $N > 3$ are not unlike the case $N = 3$.

Bivariate examples In higher dimensional spaces the sets Q can be quite interesting. We will limit ourselves to several standard examples in $\mathbb{R}^2$

First consider $A = 2I$ where I is the 2×2 identity matrix. If the members of $\mathcal{K}_A$ are described by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

then Q is simply the unit square $[0, 1]^2$ and χ_Q is the scaling function for the multiresolution analysis which it generates. However if the members of $\mathcal{K}_A$ are described by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 1 \end{pmatrix} \tag{19}$$

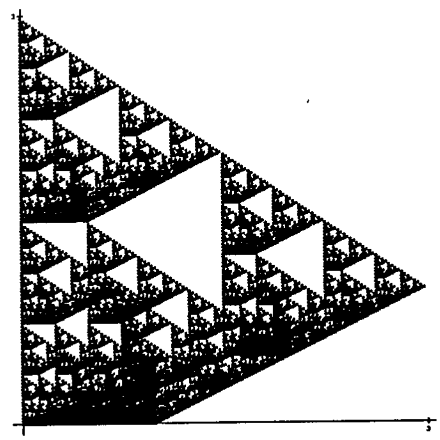

Figure 4: The set Q corresponding to $A = 2I$ and $\mathcal{K}_A$ described by the columns of (19)

then Q is set described by the shaded area in Figure 4.

If

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and the members of $\mathcal{K}_A$ are described by the columns of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then Q is the so-called twin dragon of appropriate size and position which is described by the shaded area in Figure 5.

For more examples see [17, ?, 30].

2.5.2 Univariate piecewise polynomial splines

This example nicely illustrates the usefulness of condition A6_1.

Piecewise linear splines First consider the sequence of closed subspaces of $L^2(\mathbb{R})$ defined as follows:

$$V_j = \{f \in L^2(\mathbb{R}) : f \text{ is continuous on } \mathbb{R} \text{ and linear on the intervals } [k2^{-j}, (k+1)2^{-j}],\ \ k \in \mathbb{Z}\}\ .$$

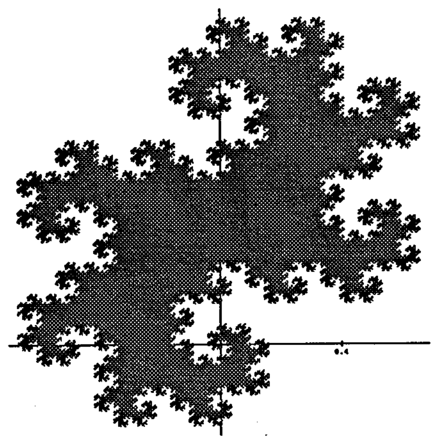

Figure 5: The set Q corresponding to the "twin dragon"

The sequence $\mathcal{V} = \{V_j\}_{j \in \mathbf{Z}}$ is a family of closed subspaces of $L^2(I\!R)$ which satisfies conditions A1 through A5 of a dyadic multiresolution analysis. That condition A6 is satisfied is not immediately clear. However it is fairly transparent that condition $A6_1$ is satisfied. Namely, integer translates of the function

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

constitute a Riesz basis for V_0. Thus $\mathcal{V}$ is a dyadic multiresolution analysis. The scaling function whose recipe was given in Subsection 2.3 is

$$\hat{\phi}_0(\xi) = \frac{\xi^{-2}}{\sqrt{\sum_{k \in \mathbf{Z}^n} |\xi + 2\pi k|^{-4}}}$$

and the scaling sequence is the sequence of coefficients $\{s_k\}_{k \in \mathbf{Z}}$ of

$$S(\xi) = \left\{ \frac{\sum_{k \in \mathbf{Z}} |\xi + 4\pi k|^{-4}}{\sum_{k \in \mathbf{Z}} |\xi + 2\pi k|^{-4}} \right\}^{1/2} = \frac{1}{2} \sum_{k \in \mathbf{Z}} s_k e^{-ik\xi} \ .$$

Note that because of the analyticity of the above expressions for $\hat{\phi}_0$ and $S(\xi)$ both $\phi_0(x)$ and $\{s_k\}_{k \in \mathbf{Z}}$ enjoy exponential decay as x and k go to $\pm\infty$ respectively. See Figure 6.

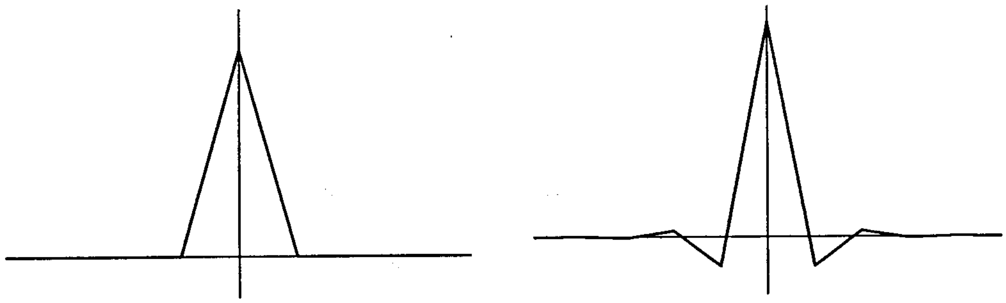

Figure 6: The piecewise linear pseudo-scaling function ϕ and the scaling function ϕ_0

The general case More generally we may consider the k fold convolution of χ, the characteristic function of the interval $[0,1]$, with itself

$$\phi(x) = \underbrace{\chi * \cdots * \chi}_{k}(x) \, .$$

This function may also be described via its Fourier transform as

$$\hat{\phi}(\xi) = \left\{ \frac{e^{-i\xi} - 1}{-i\xi} \right\}^k \, .$$

Note that ϕ is $k-2$ times continuously differentiable and coincides with a polynomial of degree $\leq k-1$ on every interval $(k, k+1)$, $k \in \mathbb{Z}$. Also observe that in the case $k = 1$ the function ϕ is simply χ and in the case $k = 2$ the function ϕ is same as the function described by (20) translated one unit to the right.

If V_0 is the $L^2(\mathbb{R})$ closure of the linear span of $\{\phi(x-m)\}_{m \in \mathbb{Z}}$ then the collection of spaces $\mathcal{V} = \{V_j\}_{j \in \mathbb{Z}}$ defined by

$$V_j = \{ f(x) \; : \; f(2^{-j}x) \in V_0 \}$$

is a dyadic multiresolution analysis of $L^2(\mathbb{R})$. As in the case of the mutiresolution analyses generated by tiles, this is a consequence of the fact that ϕ satisfies

$$\phi(x) = \frac{1}{2^{k-1}} \sum_{m=0}^{k} \binom{k}{m} \phi(2x - m)$$

and the fact that

$$\sum_{l\in\mathbf{Z}} |\hat{\phi}(\xi - 2\pi l)|^2 > 0 \qquad \text{a.e.}$$

This multiresolution analysis consists of piecewise polynomial splines of order k.

It should be mentioned that the term spline is used here for traditional reasons. In the case that k is an even integer $\mathcal{V}$ is a multiresolution analysis composed of splines in the sense of Meyer whereas in the case that k is an odd integer $\mathcal{V}$ fails to be such a multiresolution analysis.

The subspace V_0 is the intersection of $L^2(\mathbb{R})$ with the class of those tempered distributions s whose k-th order derivative is a distribution of order zero supported on $\mathbb{Z}$, in other words, s satisfies

$$\frac{d^k}{dx^k}s(x) = \sum_{m\in\mathbf{Z}} c_m \delta(x-m)$$

where $\delta(x)$ is the unit Dirac measure at the origin. This description of V_0 can, of course, be used as its definition.

As in the piecewise linear case, when $k \geq 2$ the function ϕ fails to be a scaling function. On the other hand the collection $\{\phi(x-l)\}_{l\in\mathbf{Z}}$ is a Riesz basis for V_0, and the function ϕ_0 defined by the formula for its Fourier transform

$$\hat{\phi}_0(\xi) = \frac{\xi^{-k}}{\sqrt{\sum_{l\in\mathbf{Z}} |\xi + 2\pi l|^{-2k}}}$$

is a scaling function. More details concerning this multiresolution analysis may be found in [5, 14, 33].

2.5.3 Multivariate analogues of the univariate spline examples

There are many generalizations of the examples in the previous subsection. Three classes of multivariate analogues which, in a certain sense, reduce to the univariate splines considered above are the (i) box splines, see [22, 25, 34, 38], the so-called (ii) polyharmonic splines, see [28], and (iii) the multiresolution analyses generated by the k fold convolutional iterates of the characteristic functions of the sets described in Subsection 2.5.1, see [39]. Here we indicate a few details concerning the last two classes.

Polyharmonic splines We begin with the basic setup.

Let SH_k, $k = 1, 2, \ldots$, be the class of those tempered distributions s whose k-th order Laplacian is a distribution of order zero supported on $\mathbb{Z}^n$; in other words, elements s in SH_k satisfy

$$\Delta^k s(x) = \sum_{m \in \mathbb{Z}^n} c_m \delta(x - m)$$

where Δ is the Laplacian in $\mathbb{R}^n$ and if $k \geq 2$ then $\Delta^k = \Delta\Delta^{k-1}$.

Next we recall that a matrix A on $\mathbb{R}^n$ is said to be a *similarity* if $A = \rho A_0$ where A_0 is an orthogonal matrix and ρ is a real number. If A is also an acceptable dilation for $\mathbb{Z}^n$ then, of course, $|\det A| = |\rho^n|$ is an integer ≥ 2. In the case $n = 2$ these matrices are all of the form

$$\begin{pmatrix} m & -p \\ p & m \end{pmatrix} \text{ or } \begin{pmatrix} m & p \\ p & -m \end{pmatrix}$$

where m and p are integers such that $m^2 + p^2 \geq 2$.

Finally let

$$V_0 = SH_k \bigcap L^2(\mathbb{R}^n)$$

and note that V_0 is a closed subspace of $L^2(\mathbb{R}^n)$ which is not $\{0\}$ only when $4k > n$. Furthermore if $4k > n$ the collection $\{\phi(x-m)\}_{m \in \mathbb{Z}^n}$, where function ϕ is defined by

$$\hat{\phi}(\xi) = \frac{|\xi|^{-2k}}{\{\sum_{m \in \mathbb{Z}^n} |\xi - 2\pi m|^{-4k}\}^{1/2}},$$

is an orthonormal basis for V_0.

Now, if A is both a similarity and an acceptable dilation for $\mathbb{Z}^n$ and $B = A^*$ then

(21) $$\phi(x) = \sum_{m \in \mathbb{Z}^n} s_m \phi(Ax - k)$$

whose Fourier transform is

$$\hat{\phi}(\xi) = S(B^{-1}\xi)\hat{\phi}(B^{-1}\xi)$$

where

$$S(\xi) = \left\{ \frac{\sum_{m \in \mathbb{Z}^n} |\xi + 2\pi Bm|^{-4k}}{\sum_{m \in \mathbb{Z}^n} |\xi + 2\pi m|^{-4k}} \right\}^{1/2} = \frac{1}{a} \sum_{m \in \mathbb{Z}^n} s_k e^{-i\langle m, \xi \rangle} .$$

Note that because of the analyticity of the above expressions for $\hat{\phi}$ and $S(\xi)$ both $\phi(x)$ and $\{s_m\}_{m\in\mathbb{Z}^n}$ enjoy exponential decay as $|x|$ and $|m|$ go to ∞ respectively.

In view of (21) and the fact that $\{\phi(x-m)\}_{m\in\mathbb{Z}^n}$ is an orthonormal basis for V_0 it follows that, if A is both a similarity and an acceptable dilation for $\mathbb{Z}^n$, then the collection of spaces $\mathcal{V} = \{V_j\}_{j\in\mathbb{Z}}$ defined by

$$\begin{aligned} V_j &= U_A^{-j} V_0 \\ &= \{f(x) \,:\, f(A^{-j}x) \in V_0\} \end{aligned}$$

is a multiresolution analysis associated to $(\mathbb{Z}^n, A)$. If $2k > n$ then this multiresolution analysis is composed of splines in the sense of Meyer. For more details see [28].

Note that in the case $n = 1$ these classes reduce to the classes of piecewise polynomial splines considered in the previous subsection which are of even order.

Multiresolution analyses generated by the k fold convolutional iterates of certain characteristic functions The characteristic functions are of sets which were considered in Subsection 2.5.1. The basic setup is the following:

Suppose k is a positive integer, A is an acceptable dilation for $\mathbb{Z}^n$, $\mathcal{K}_A$ is a full collection of representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$, and Q is the compact set defined by (16). Let ϕ be the function defined by

$$\phi(x) = \underbrace{\chi * \cdots * \chi}_{k}(x)$$

where χ is the characteristic function of Q. This function may also be described via its Fourier transform as

$$\hat{\phi}(\xi) = \{\hat{\chi}(\xi)\}^k \, .$$

Note that

$$\phi(x) = \sum_{m\in\mathbb{Z}^n} s_m \phi(Ax - k) \qquad (22)$$

whose Fourier transform is

$$\hat{\phi}(\xi) = S(B^{-1}\xi)\hat{\phi}(B^{-1}\xi)$$

where $B = A^*$ and

$$S(\xi) = \left\{ \frac{1}{a} \sum_{\nu \in \mathcal{K}_A} e^{-i\langle \nu, \xi \rangle} \right\}^k = \frac{1}{a} \sum_{m \in \mathbb{Z}^n} s_k e^{-i\langle m, \xi \rangle} .$$

The function $\phi(x)$ has compact support and the sequence $\{s_m\}_{m \in \mathbb{Z}^n}$ has only a finite number of non-zero terms.

As in Subsection 2.5.1 the function ϕ generates a multiresolution analysis in the following sense: If V_0 is the $L^2(\mathbb{R}^n)$ closure of the linear span of $\{\phi(x-m)\}_{m \in \mathbb{Z}^n}$ then the collection of spaces $\mathcal{V} = \{V_j\}_{j \in \mathbb{Z}}$ defined by

$$\begin{aligned} V_j &= U_A^{-j} V_0 \\ &= \{f(x) \ : \ f(A^{-j}x) \in V_0\} \end{aligned}$$

is a multiresolution analysis associated to $(\mathbb{Z}^n, A)$. This is a consequence of the fact that ϕ satisfies (22) and the fact that

$$\sum_{m \in \mathbb{Z}^n} |\hat{\phi}(\xi - 2\pi m)|^2 > 0 \qquad \text{a.e.}$$

Furthermore, if $|Q| = 1$ then $\{\phi(x-m)\}_{m \in \mathbb{Z}^n}$ is a Riesz basis for V_0. On the other hand if $|Q| > 1$ then $\{\phi(x-m)\}_{m \in \mathbb{Z}^n}$ fails to be a Riesz basis for V_0. Nevertheless in either case it is not difficult to find a scaling function for $\mathcal{V}$. The function ϕ_0 defined by the formula for its Fourier transform

$$\hat{\phi}_0(\xi) = \frac{\hat{\phi}(\xi)}{\left\{ \sum_{m \in \mathbb{Z}^n} |\hat{\phi}(\xi - 2\pi m)|^2 \right\}^{1/2}}$$

does the job in both cases. Unfortunately ϕ_0 does not in general enjoy compact support. Other interesting facts concerning these multiresolution analyses can be found in [39].

Note that in the case $n = 1$ and appropriate choice of $\mathcal{K}_A$ these classes reduce to the classes of piecewise the polynomial splines considered in the previous subsection.

2.5.4 Compactly supported scaling functions

Examples of compactly supported scaling functions can be found in Subsection 2.5.1. The fact that the spline examples considered above do not give rise to compactly supported scaling functions suggests that examples which are smoother than those considered in Subsection 2.5.1 may be more difficult to construct. Indeed the construction of such examples in the univariate case is the central topic of Daubechies work [12], see also [14].

The simplest example considered in [12] involves the function ϕ described in Example 1.2.3 and Figure 2. This function generates a dyadic multiresolution analysis of $L^2(I\!\!R)$ and is also a scaling function for this multiresolution analysis. In other words, if V_0 is the $L^2(I\!\!R)$ closure of the linear span of $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ then the collection of spaces $\mathcal{V} = \{V_j\}_{j\in\mathbb{Z}}$ defined by

$$V_j = \{f(x) \ : \ f(2^{-j}x) \in V_0\}$$

is a dyadic multiresolution analysis and ϕ is a scaling function for this multiresolution analysis.

Recipes for constructing dyadic multiresolution analyses of $L^2(I\!\!R)$ with compactly supported scaling functions which enjoy any desired finite order of smoothness can be found in [14] along with many specific examples. The rough idea is to construct appropriate scaling sequences and define the corresponding scaling functions as the compactly supported solutions of the corresponding two scale difference equations. The difficult part is guaranteeing that the resulting solutions are indeed scaling functions with the desired properties.

Multivariate examples can be constructed by taking appropriate tensor products of univariate examples. For example, if ϕ_1 and ϕ_2 are univariate scaling functions for multiresolution analyses $\mathcal{V}_1$ and $\mathcal{V}_2$ which are associated to the dilations $x \longrightarrow N_1x$ and $x \longrightarrow N_2x$ respectively, ϕ_1 may or may not be the same as ϕ_2, then the bivariate function ϕ defined by

$$\phi(x,y) = \phi_1(x)\phi_2(y)$$

is a scaling function for a multiresolution analysis of $L^2(I\!\!R^2)$ associated to $(\mathbb{Z}^2, A)$ where

$$A = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} .$$

On the other hand, in the case of a general acceptable dilation A in $I\!\!R^n$, $n \geq 2$, I am not presently aware of any recipes for the construction of compactly supported scaling functions for multiresolution analyses associated to $(\mathbb{Z}^n, A)$ which enjoy an arbitrarily high finite order of smoothness. Some work in this direction may be found in [8].

2.5.5 Band limited scaling functions

Suppose Ω is a compact subset of $I\!R^n$ which enjoys the following properties:

- $\Omega \subset B\Omega$.
- $|\Omega \bigcap \{\Omega + 2\pi k\}| = 0$ for any element k in $Z\!\!Z^n \setminus \{0\}$.
- $$\bigcup_{k \in Z^n} \{\Omega + 2\pi k\} \simeq I\!R^n \, .$$
- For all cubes Q of finite diameter in $I\!R^n$
$$\lim_{j\to\infty} \frac{1}{|B^{-j}Q|} \int_{B^{-j}Q} \chi_\Omega(\xi) d\xi = 1 \, .$$

Consider the collection of subspaces $\mathcal{V} = \{V_j\}_{j\in \mathbb{Z}}$ defined by

$$V_j = \{f \in L^2(I\!R^n) : \operatorname{supp} \hat{f} \subset B^j\Omega\} \, .$$

The properties of Ω imply that $\mathcal{V}$ is a multiresolution analysis associated to $(Z\!\!Z^n, A)$. The function ϕ defined by

(23) $$\hat{\phi}(\xi) = \chi_\Omega(\xi)$$

is a scaling function for this multiresolution analysis.

Note that ϕ is not only a scaling function but also a *cardinal function* for V_0. Namely,

$$\phi(k) = \delta_{0,k}$$

for all k in $Z\!\!Z^n$ and every f in V_0 enjoys the representation

$$f(x) = \sum_{k\in \mathbb{Z}^n} f(k)\phi(x-k) \, .$$

Since the last equation is also the expansion of f with respect to the orthogonal basis $\{\phi(x-k)\}_{k\in\mathbb{Z}^n}$ of V_0 the multiresolution analysis $\mathcal{V}$ is composed of splines in the sence of Meyer.

Unfortunately $\phi(x)$ has poor decay properties as $|x|$ goes to infinity. In particular, $|x|\phi(x)$ fails to be in $L^2(I\!R^n)$.

The simplest example is the case where $n = 1$, Ω is the interval $[-\pi, \pi]$, and $A = B = 2$. Here $\phi(x) = \operatorname{sinc} x$ and we refer to this case as the sinc multiresolution analysis.

Examples with better decay properties Suppose that in addition to the above listed properties Ω satisfies the following:

- The interior of $B\Omega$ contains Ω.

In this case there exit functions which are infinitely differentiable, are identically equal to 1 on Ω, and have support in $B\Omega$. Choose one which is non-negative, call it h, and define ϕ_0 via

$$\hat{\phi}(\xi) = h(\xi) \, .$$

Note that $\{\phi_0(x-k)\}_{k\in\mathbf{Z}^n}$ is a Riesz basis for the closure of its linear span V_0. If

$$V_j = \{f(x) : \; f(A^{-j}x) \in V_0\}$$

then $\mathcal{V} = \{V_j\}_{j\in\mathbf{Z}}$ is a multiresolution anlysis associated to $(\mathbb{Z}^n, A)$ with a scaling function ϕ defined by

$$\hat{\phi}(\xi) = \frac{h(\xi)}{\{\sum_{k\in\mathbf{Z}^n} |h(\xi - 2\pi k)|^2\}^{1/2}} \, .$$

Unlike the earlier case, ϕ is not a cardinal function for V_0. Nevertheless $\mathcal{V}$ is composed of spline functions in the sense of *Meyer*; the cardinal function λ in this case is given by

$$\hat{\lambda}(\xi) = \frac{h(\xi)}{\sum_{k\in\mathbf{Z}^n} h(\xi - 2\pi k)} \, .$$

Furthermore both $\phi(x)$ and $\lambda(x)$ decay faster than the reciprocal of any polynomial as $|x|$ goes to infinity; in other word, for any positive integer p both

$$|x|^p \phi(x) \qquad \text{and} \qquad |x|^p \lambda(x)$$

are bounded on $\mathbb{R}^n$.

Unfortunately, even in the simplest cases, an explicit formula for either ϕ or λ is not easily available. For this reason in certain applications one may wish to sacrifice some of the decay properties by choosing less regular h.

3 Wavelets

3.1 Introduction

In what follows A is an acceptable dilation for $\mathbb{Z}^n$, $\mathcal{V} = \{V_j\}_{j \in \mathbf{Z}}$ is multiresolution analysis of $L^2(\mathbb{R}^n)$ associated with $(\mathbb{Z}^n, A)$, and ϕ is a scaling function for $\mathcal{V} = \{V_j\}_{j \in \mathbf{Z}}$ with corresponding scaling sequence $\{s_j\}_{j \in \mathbf{Z}^n}$. Let $b = a - 1$ where $a = |\det A|$. The objective of Subsections 3.1-3.5 is to outline how this framework gives rise to a wavelet basis and, in the process, to constructively prove the following:

Theorem 1 *There exist b functions $\psi_1, \ldots, \psi_b$ which enjoy the following properties:*

- *If $\psi_0 = \phi$ is a scaling function for $\mathcal{V}$ then the collection*

$$\{\psi_j(x - k)\}_{j \in \{0,1,\ldots,b\},\, k \in \mathbf{Z}^n}$$

  is a complete orthonormal system for the subspace V_1.

- *The collection*

$$\{a^{l/2}\psi_j(A^l x - k)\}_{j \in \{1,\ldots,b\},\, k \in \mathbf{Z}^n,\, l \in \mathbf{Z}}$$

  is a complete orthonormal system for $L^2(\mathbb{R}^n)$.

Remarks The functions $\psi_1, \ldots, \psi_b$ are a collection of fundamental wavelets associated with the multiresolution analysis $\mathcal{V}$. The development below will show (i) that this collection is not unique in any sense, although in the case $a = 2$ there is in some sense a canonical formula for obtaining ψ_1 from the scaling function ψ_0, and (ii) that b is the minimal number in this collection.

An important objective is the construction of wavelets $\psi_1, \ldots, \psi_b$ which have the same decay, for large $|x|$, that the "best" scaling function has. For example if $\mathcal{V}$ has a compactly supported scaling function then one would like the ψ's to have compact support. The construction outlined below does not in general guarantee this. However, because of the importance of this matter, we will make remarks concerning this issue at convenient places in the development.

3.2 The basic setup

Let W_j be the orthogonal complement of V_j in V_{j+1}. In other words

$$V_{j+1} = W_j \oplus V_j \tag{24}$$

where $\oplus$ denotes the orthogonal direct sum of linear subspaces. Iterating (24) results in

$$V_{j+1} = W_j \oplus W_{j-1} \oplus \cdots \oplus W_{j-k} \oplus V_{j-k}$$

for any positive integer k The last identity together with properties A2 and A3 imply that

$$L^2(I\!R^n) = \sum_{j \in \mathbb{Z}} \oplus W_j \tag{25}$$

Now, the fact that mapping U_A defined by $U_A f(x) = a^{-1/2} f(A^{-1}x)$ is a unitary transformation together with property A5 imply that

$$W_j = U_A^{-j} W_0 \tag{26}$$

for all integers j. Thus knowledge of the structure of the subspace W_0 gives us analogous knowledge of W_j for all j. In particular, if there are functions $\psi_1, \ldots, \psi_b$ such that

$$\{\psi_i(x-k)\}_{i \in \{1,\ldots,b\}, k \in \mathbb{Z}^n} \tag{27}$$

is a complete orthonormal system for W_0 then

$$\{a^{j/2}\psi_i(A^j x - k)\}_{i \in \{1,\ldots,b\}, j \in \mathbb{Z}^n}$$

is a complete orthonormal system for W_j.

In what follows we will constructively show that there are $b = a - 1$ functions $\psi_1, \ldots, \psi_b$ whose $Z\!\!Z^n$ translates (27) are a complete orthonormal system for W_0. In view of (25) and (26) we may conclude that

$$\{a^{j/2}\psi_i(A^j x - k)\}_{i \in \{1,\ldots,b\}, j \in \mathbb{Z}, k \in \mathbb{Z}^n} \tag{28}$$

is a wavelet basis for $L^2(I\!R^n)$.

Since $W_0 \subset V_1$ it follows that the fundamental wavelets $\psi_1, \ldots, \psi_b$ satisfy

$$\psi_i(x) = \sum_{k \in \mathbb{Z}^n} s_{i,k}\phi(Ax - k) \tag{29}$$

for $i \in \{1, \ldots, b\}$ where ϕ is a scaling function for the multiresolution analysis $\mathcal{V}$ and $\{s_{i,k}\}_{k\in\mathbb{Z}^n}$ are appropriate sequences in $l^2(\mathbb{Z}^n)$. Now, if $\{s_{0,k}\}_{k\in\mathbb{Z}^n} = \{s_k\}_{k\in\mathbb{Z}^n}$ is the scaling sequence for ϕ, namely

$$\phi(x) = \sum_{k\in\mathbb{Z}^n} s_{0,k}\phi(Ax-k) \ ,$$

then from the orthogonality relations

$$\int_{\mathbb{R}^n} \psi_{i_1}(x-k_1)\overline{\psi_{i_2}(x-k_2)}dx = \delta_{i_1,i_2}\delta_{k_1,k_2}$$

where $\psi_0 = \phi$, i_1 and i_2 are indicies taking values in $\{0, \ldots, b\}$, and k_1 and k_2 are elements in $\mathbb{Z}^n$, we may conclude that the sequences $\{s_{i,k}\}_{k\in\mathbb{Z}^n}$, $i = 0, 1, \ldots, m$, must satisfy

$$\sum_{k\in\mathbb{Z}^n} s_{i_1,k-Aj}\overline{s_{i_2,k}} = a\delta_{i_1,i_2}\delta_{0,j} \tag{30}$$

for $j \in \mathbb{Z}^n$.

Conversely, if $\{s_{i,k}\}_{k\in\mathbb{Z}^n}$, $i = 1, \ldots, m$, are sequences in $l^2(\mathbb{Z}^n)$ which satisfy (30) where if $\{s_{0,k}\}_{k\in\mathbb{Z}^n}$ is the scaling sequence for ϕ then the functions $\psi_1, \ldots, \psi_m$ defined via (29) are candidates for fundamental wavelets. Indeed the constructive proof of the existence of a collection of fundamental wavelets essentially relies on the construction of such sequences. The details however are carried out in the Fourier transform or frequency domain.

3.3 A characterization of W_0

Recall that ϕ is a scaling function for $\mathcal{V} = \{V_j\}_{j\in\mathbb{Z}}$ with corresponding scaling sequence $\{s_j\}_{j\in\mathbb{Z}^n}$ and

$$\hat{\phi}(\xi) = S(B^{-1}\xi)\hat{\phi}(B^{-1}\xi)$$

where $B = A^*$ is the adjoint of A and $S(\xi)$ is the $2\pi Z^n$ periodic function

$$S(\xi) = \frac{1}{a}\sum_{k\in Z^n} s_k e^{-i\langle k,\xi\rangle}.$$

Also recall that $\mathcal{K}_A$ and $\mathcal{K}_B$ denote full collections of representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$ and $\mathbb{Z}^n/B\mathbb{Z}^n$ respectively. See Subsection 2.4.

Note that every $2\pi B\mathbb{Z}^n$ periodic function is of the form $F(B^{-1}\xi)$ where $F(\xi)$ is $2\pi\mathbb{Z}^n$ periodic.

Proposition 1 *The function f is in W_0 if and only if f enjoys the representation*

$$\hat{f}(\xi) = F(B^{-1}\xi)\hat{\phi}(B^{-1}\xi) \tag{31}$$

where $F(\xi)$ is in $L^2(\mathbb{R}^n/2\pi\mathbb{Z}^n)$ and satisfies

$$\sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu))\overline{S(B^{-1}(\xi - 2\pi\nu))} = 0 \tag{32}$$

for almost all ξ.

Proof: Observe that every element g in V_0 enjoys the representation

$$\hat{g}(\xi) = G(\xi)\hat{\phi}(\xi) \tag{33}$$

where $G(\xi)$ is in $L^2(\mathbb{R}^n/2\pi\mathbb{Z}^n)$ and, conversely, every such G gives rise to an element g in V_0 via (33). Since $\hat{g}(\xi)$ is in $\widehat{V}_0$ if and only if $\hat{g}(B^{-1}\xi)$ is in $\widehat{V}_1$ and $W_0 \subset V_1$ we may conclude that reprsentation (31) follows from (33).

Relation (32) is equivalent to fact that f is orthogonal to V_0. To see this use the following:

- Plancherel's formula,
- the Fourier transform of the scaling relation (11),
- the periodization trick (9), and
- the fact that if $H(\xi)$ is a $2\pi\mathbb{Z}^n$ periodic function then

  $$\sum_{j \in \mathbb{Z}^n} H(B^{-1}(\xi - 2\pi j)|\hat{\phi}(B^{-1}(\xi - 2\pi j)|^2 = \sum_{\nu \in \mathcal{K}_B} H(B^{-1}(\xi - 2\pi\nu)) \, .$$

  This last item follows from (5), the periodicity of H, and an application of (4).

Explicitly, for every g in V_0 write

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} F(B^{-1}\xi)\hat{\phi}(B^{-1}\xi)\overline{G(\xi)\hat{\phi}(\xi)}d\xi$$

$$= \int_{\mathbb{R}^n} F(B^{-1}\xi)\overline{S(B^{-1}\xi)G(\xi)}|\hat{\phi}(B^{-1}\xi)|^2 d\xi$$

$$= \int_{Q_\pi} \left\{ \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu))\overline{S(B^{-1}(\xi - 2\pi\nu))} \right\} \overline{G(B\xi)} d\xi$$

or, more succinctly,

$$(34) \qquad \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx =$$

$$\int_{Q_\pi} \left\{ \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu))\overline{S(B^{-1}(\xi - 2\pi\nu))} \right\} \overline{G(B\xi)} d\xi \,.$$

Thus, if (32) holds then (34) implies that f is orthogonal to all g in V_0. Conversely, if f is orthogonal to all g in V_0 the integral on the right hand side of (34) must vanish for all G in $L^2(\mathbb{R}^n/2\pi\mathbb{Z}^n)$ which implies (32). ∎

3.3.1 An alternate expression for $\sum F(B^{-1}(\xi - 2\pi\nu))\overline{S(B^{-1}(\xi - 2\pi\nu)}$

The $2\pi\mathbb{Z}^n$ periodization, or average over the group $\mathbb{Z}^n/B\mathbb{Z}^n$, of

$$F(B^{-1}\xi)\overline{S(B^{-1}\xi)}$$

on the left hand side of formula (32) suggests a kind of inner product which maps pairs of $2\pi B\mathbb{Z}^n$ periodic functions into a $2\pi\mathbb{Z}^n$ periodic function. This "product" may be more conveniently expressed in terms of the $2\pi\mathbb{Z}^n$ periodic components of the respective $2\pi B\mathbb{Z}^n$ periodic functions. To wit:

Every $2\pi B\mathbb{Z}^n$ periodic function $F(B^{-1}\xi)$ enjoys the representation

$$(35) \qquad F(B^{-1}\xi) = \frac{1}{\sqrt{a}} \sum_{\kappa \in \mathcal{K}_A} F_\kappa(\xi) e^{-i\langle A^{-1}\kappa, \xi\rangle}$$

where $F_\kappa(\xi)$, $\kappa \in \mathcal{K}_A$, is a $2\pi\mathbb{Z}^n$ periodic function which can be derived from $F(B^{-1}\xi)$ via the formula

$$(36) \qquad F_\kappa(\xi) = \frac{1}{\sqrt{a}} \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) e^{\langle A^{-1}\kappa, \xi - 2\pi\nu\rangle} \,.$$

If $G(B^{-1}\xi)$ is another $2\pi B\mathbb{Z}^n$ periodic function then

$$(37) \qquad \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu))\overline{G(B^{-1}(\xi - 2\pi\nu))} = \sum_{\kappa \in \mathcal{K}_A} F_\kappa(\xi)\overline{G_\kappa(\xi)}$$

where, of course, G_κ is related to G in the same way that F_κ is related to F. Note that (37) gives us an alternate expression for the inner product defined by the right hand side of (32).

Formulas (32), (35), and (37) suggest that $2\pi B\mathbb{Z}^n$ periodic functions $F(B^{-1}\xi)$ be viewed as a tuples of $2\pi\mathbb{Z}^n$ periodic functions

$$\vec{F}(\xi) = (F_{\kappa_1}(\xi), \dots F_{\kappa_a}(\xi))$$

where $(\kappa_1, \dots, \kappa_a)$ is an ordering of $\mathcal{K}_A$ and $F_{\kappa_1}, \dots, F_{\kappa_a}$ are related to $F(B^{-1}\xi)$ via (36). We do this in what follows.

3.3.2 Details

We begin by noting the following:

Lemma 1 *If $\nu \in \mathcal{K}_B$ then*

$$\sum_{\kappa\in\mathcal{K}_A} e^{i2\pi\langle A^{-1}\kappa,\nu\rangle} = a\delta_{0,\nu} \quad . \tag{38}$$

This is nothing more than the observation that the mapping

$$\kappa + A\mathbb{Z}^n \longrightarrow e^{i2\pi\langle A^{-1}\kappa,\nu\rangle}$$

is a character of the coset group $\mathbb{Z}^n/A\mathbb{Z}^n$ and (38) is a well known relation for such functions. For example see [36]. For the sake of completeness we recall the argument.

Proof: If $\nu = 0$ then (38) is transparent. If $\nu \neq 0$ then there is a μ in $\mathcal{K}_A$ such that $e^{i2\pi\langle A^{-1}\mu,\nu\rangle} \neq 1$. Since $\mathcal{K}_A - \mu$ is also a full collection of coset representatives so that

$$\sum_{\kappa\in\mathcal{K}_A} e^{i2\pi\langle A^{-1}(\kappa-\mu),\nu\rangle} = \sum_{\kappa\in\mathcal{K}_A} e^{i2\pi\langle A^{-1}\kappa,\nu\rangle} \quad ,$$

we may write

$$\begin{aligned} \textstyle\sum_{\kappa\in\mathcal{K}_A} e^{i2\pi\langle A^{-1}\kappa,\nu\rangle} &= e^{i2\pi\langle A^{-1}(\mu,\nu\rangle} \textstyle\sum_{\kappa\in\mathcal{K}_A} e^{i2\pi\langle A^{-1}(\kappa-\mu),\nu\rangle} \\ &= e^{i2\pi\langle A^{-1}\mu,\nu\rangle} \textstyle\sum_{\kappa\in\mathcal{K}_A} e^{i2\pi\langle A^{-1}\kappa,\nu\rangle} \quad . \end{aligned}$$

Since $e^{i2\pi\langle A^{-1}\mu,\nu\rangle} \neq 1$ the last string of equalities implies (38). ■

Formulas (35) and (37) are easy consequences of this lemma. To see (35) use identity (36) for F_κ, interchange order of summation, and apply formula (38) to write

$$\frac{1}{\sqrt{a}} \sum_{\kappa \in \mathcal{K}_A} F_\kappa(\xi) e^{-i\langle A^{-1}\kappa, \xi\rangle}$$

$$= \frac{1}{\sqrt{a}} \sum_{\kappa \in \mathcal{K}_A} \left\{ \frac{1}{\sqrt{a}} \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) e^{\langle A^{-1}\kappa, \xi - 2\pi\nu\rangle} \right\} e^{-i\langle A^{-1}\kappa, \xi\rangle}$$

$$= \frac{1}{a} \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) \left\{ \sum_{\kappa \in \mathcal{K}_A} e^{i2\pi\langle A^{-1}\kappa, \nu\rangle} \right\} = F(B^{-1}\xi) \ .$$

To see (37) use formula (36) for F_κ and G_κ, interchange various orders of summation appropriately, and use identity (38) to write

$$\sum_{\kappa \in \mathcal{K}_A} F_\kappa(\xi) \overline{G_\kappa(\xi)}$$

$$= \sum_{\kappa \in \mathcal{K}_A} \left\{ \frac{1}{\sqrt{a}} \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) e^{-i\langle A^{-1}\kappa, \xi - 2\pi\nu\rangle} \right\} \times$$

$$\left\{ \frac{1}{\sqrt{a}} \sum_{\mu \in \mathcal{K}_B} \overline{G(B^{-1}(\xi - 2\pi\mu))} e^{\langle A^{-1}\kappa, \xi - 2\pi\mu\rangle} \right\}$$

$$= \sum_{\nu \in \mathcal{K}_B} \sum_{\mu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) \overline{G(B^{-1}(\xi - 2\pi\mu))} \times$$

$$\left\{ \frac{1}{a} \sum_{\kappa \in \mathcal{K}_A} e^{-i2\pi\langle A^{-1}\kappa, \nu - \mu\rangle} \right\}$$

$$= \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) \overline{G(B^{-1}(\xi - 2\pi\nu))}$$

Remark We alert the reader familiar with the application of these formulas in [25] that a slightly different normalization, which results in a factor of a in the right hand side of formula (37), was used there.

3.4 Consequences of the characterization of W_0

In what follows we use the following conventions:

- $a = |\det A|$ and $b = a - 1$
- $\{\kappa_0, \ldots, \kappa_b\}$ and $\{\nu_0, \ldots, \nu_b\}$ are fixed orderings of $\mathcal{K}_A$ and $\mathcal{K}_B$ respectively with $\kappa_0 = \nu_0 = 0$.
- If $F(B^{-1}\xi)$ is a $2\pi\mathbb{Z}^n$ periodic function then $\vec{F}(\xi) = (F_0(\xi), \ldots, F_b(\xi))$ where
$$F_l(\xi) = \frac{1}{\sqrt{a}} \sum_{\nu \in \mathcal{K}_B} F(B^{-1}(\xi - 2\pi\nu)) e^{\langle A^{-1}\kappa_l, \xi - 2\pi\nu \rangle}$$
for $l \in \{0, \ldots, b\}$. We identify $\vec{F}(\xi)$ with $F(B^{-1}\xi)$ as indicated in the previous subsection, namely,
$$F(B^{-1}\xi) = \frac{1}{\sqrt{a}} \sum_{l=0}^{b} F_l(\xi) e^{-i\langle A^{-1}\kappa_l, \xi \rangle} \quad . \tag{39}$$
- If $F(B^{-1}\xi)$ and $G(B^{-1}\xi)$ are $2\pi B\mathbb{Z}^n$ periodic functions then
$$\langle \vec{F}(\xi), \vec{G}(\xi) \rangle_B = \sum_{l=0}^{b} F_l(\xi) \overline{G_l(\xi)} \quad .$$
- The class of all measurable $2\pi\mathbb{Z}^n$ periodic vector fields
$$\vec{F}(\xi) = (F_0(\xi), \ldots, F_b(\xi))$$
with values in $\mathbb{C}^a$ is denoted by $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$. The subclass consisting of those elements of $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ which are locally square integrable is denoted by $L^2(VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a))$ and for such elements
$$\langle \vec{F}, \vec{G} \rangle = \int_{Q_\pi} \langle \vec{F}(\xi), \vec{G}(\xi) \rangle_B \, d\xi \quad .$$

In view of the observations made in Subsection 3.3.1 the conventions allow us to express the identity (12) involving the scaling factor $S(B^{-1}\xi)$ and identity (32) characterizing the periodic factors in the Fourier transform of W_0 in a more convenient way. Namely, if $S(B^{-1}\xi)$ is a scaling factor then the corresponding element $\vec{S}(\xi)$ in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ must satisfy

$$(40) \qquad \langle \vec{S}(\xi), \vec{S}(\xi)\rangle_B = 1$$

and if $F(B^{-1}\xi)\hat{\phi}(\xi)$ is the Fourier transform of f in W_0 then

$$(41) \qquad \langle \vec{F}(\xi), \vec{S}(\xi)\rangle_B = 0$$

for almost all ξ. Note that identity (40) implies that $\vec{S}(\xi)$ is a $2\pi\mathbb{Z}^n$ periodic vector field with values on the unit sphere in $\mathbb{C}^a$. Relation (41) means that $\vec{F(\xi)}$ is a $2\pi\mathbb{Z}^n$ periodic vector field which is orthogonal to $\vec{S}(\xi)$.

Recall that the Fourier transform of every element in V_1 may be expressed as

$$\hat{f}(\xi) = F(B^{-1}\xi)\hat{\phi}(\xi)$$

where $F(\xi)$ is in $L^2(\mathbb{R}^n/2\pi\mathbb{Z}^n)$. In view of the identification of $F(B^{-1}\xi)$ with $\vec{F}(\xi)$ we see that every f in V_1 can be identified with a unique $\vec{F}(\xi)$ in $L^2(VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a))$ in a natural way.

Suppose $\vec{X}_0, \ldots, \vec{X}_b$ is a basis for $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ in the sense that the Gramm matrix

$$(42) \qquad \mathbf{Gr}(\vec{X}_0, \ldots, \vec{X}_b) = (\langle \vec{X}_l(\xi), \vec{X}_m(\xi)\rangle_B)_{l,m=0,\ldots,b}$$

is both uniformly bounded and uniformly strictly positive definite, that is, there are positive constants c and C such that

$$(43) \qquad c\sum_{m=0}^{b} |\tau_m|^2 \leq \sum_{l=0}^{b}\sum_{m=0}^{b} \langle \vec{X}_l(\xi), \vec{X}_m(\xi)\rangle_B \tau_l \bar{\tau}_m \leq C\sum_{m=0}^{b} |\tau_m|^2$$

for almost all ξ and all complex a tuples $(\tau_0, \ldots, \tau_b)$. Then every $\vec{F}(\xi)$ in $L^2(VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a))$ can be expressed as

$$\vec{F}(\xi) = \sum_{l=0}^{b} H_l(\xi)\vec{X}_l(\xi)$$

for some uniquely determined element $\vec{H}(\xi) = (H_0(\xi), \ldots, H_b(\xi))$ in $L^2(VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a))$.

Consider the functions $\psi_0(x), \ldots, \psi_b(x)$ defined by

$$\hat{\psi}_l(\xi) = X_l(B^{-1}\xi)\hat{\phi}(B^{-1}\xi) \tag{44}$$

where $\vec{X}_0, \ldots, \vec{X}_b$ are a basis of the type discussed above. We remind the reader that ϕ is the scaling function and

$$X_l(B^{-1}\xi) = \frac{1}{\sqrt{a}} \sum_{m=0}^{b} X_{lm}(\xi) e^{-i\langle A^{-1}\kappa_m, \xi\rangle}$$

where $\vec{X}_l(\xi) = (X_{l0}(\xi), \ldots, X_{lb}(\xi))$, $l \in \{0, \ldots, b\}$. In view of the observations made in the previous paragraph the $\mathbb{Z}^n$ translates of the ψ_l's, namely

$$\{\psi_l(x-j)\}_{l\in\{0,\ldots,b\},\, j\in\mathbf{Z}^n} \quad ,$$

constitute a Riesz basis for V_1.

In paricular if

$$\langle \vec{X}_l(\xi), \vec{S}_0(\xi)\rangle_B = 0 \qquad \text{for all } \xi \text{ and} \qquad l \in \{1, \ldots, b\} \; ,$$

then

$$\{\psi_l(x-j)\}_{l\in\{1,\ldots,b\},\, j\in\mathbf{Z}^n}$$

is a Riesz basis for W_0. In analogy with the notion of pseudo-scaling function, we refer to the members of such a basis as *pseudo-wavelets* and the set $\psi_1(x), \ldots, \psi_b(x)$ as a full collection of fundamental pseudo-wavelets.

Furthermore if $\vec{X}_0 = \vec{S}$ and

$$\langle \vec{X}_l(\xi), \vec{X}_k(\xi)\rangle_B = \delta_{l,k} \qquad \text{for all } \xi \text{ and} \qquad l, k \in \{0, \ldots, b\} \; ,$$

then

$$\{\psi_l(x-j)\}_{l\in\{1,\ldots,b\},\, j\in\mathbf{Z}^n}$$

is an orthogonal basis for W_0. In other words, $\psi_1(x), \ldots, \psi_b(x)$ is a full collection of fundamental wavelets.

We summarize some of these observations as follows:

Proposition 2 *Suppose $\vec{X}_0, \ldots, \vec{X}_b$ is a basis for $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ in the sense that the Gramm matrix (42) satisfies (43). Then the $\mathbb{Z}^n$ translates of the functions $\psi_0(x), \ldots, \psi_b(x)$ defined by (44), namely the functions of x*

$$\{\psi_l(x-j)\}_{l\in\{0,\ldots,b\},\, j\in\mathbf{Z}^n} \quad ,$$

are a Riesz basis for V_1. In particular if $\vec{X}_0 = \vec{S}$ and

$$\langle \vec{X}_l(\xi), \vec{X}_k(\xi) \rangle_B = delta_{l,k} \qquad \textit{for all } \xi \textit{ and} \qquad l,k \in \{0,\ldots,b\} ,$$

then

$$\{\psi_l(x-j)\}_{l\in\{1,\ldots,b\},\, j\in \mathbb{Z}^n}$$

is an orthogonal basis for W_0 and thus $\psi_1(x),\ldots,\psi_b(x)$ is a full collection of fundamental wavelets.

3.5 A recipe

It is now clear, in principle at least, how to obtain a full collection of fundamental wavelets in terms of a scaling function $\phi(x)$ and the corresponding scaling factor $S(B^{-1}\xi)$ which are associated with the multiresolution analysis $\mathcal{V}$. The recipe goes as follows:

- First, for notational convenience, set $S_0(B^{-1}\xi) = S(B^{-1}\xi)$. Next, identify $S_0(B^{-1}\xi)$ with the vector field

  $$\vec{S}_0(\xi) = (S_{00}(\xi),\ldots,S_{0b}(\xi))$$

  in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$. The components of $\vec{S}_0(\xi)$ can be evaluated in terms of $S_0(B^{-1}\xi)$ via formula (39)

- Select b elements

  $$\vec{S}_m(\xi) = (S_{m0}(\xi),\ldots,S_{mb}(\xi)) , \qquad m \in \{1,\ldots,b\} ,$$

  in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ such that

  $$(45) \qquad \begin{pmatrix} S_{00}(\xi) & S_{01}(\xi) & \cdots & S_{0b}(\xi) \\ S_{10}(\xi) & S_{11}(\xi) & \cdots & S_{1b}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ S_{b0}(\xi) & S_{b1}(\xi) & \cdots & S_{bb}(\xi) \end{pmatrix}$$

  is a unitary matrix for all ξ. In other words, $\vec{S}_1,\ldots,\vec{S}_b$ should be chosen so that

  $$\langle \vec{S}_l(\xi), \vec{S}_m(\xi) \rangle_B = \delta_{l,m}$$

for all ξ and for all l and m in $\{0, \ldots, b\}$. This should be possible, in principle, since

$$\langle \vec{S}_0(\xi), \vec{S}_0(\xi) \rangle_B = 1$$

for all ξ.

- Define ψ_l, $l = 1, \ldots, b$, via the formula for its Fourier transform

$$\hat{\psi}_l(\xi) = S_l(B^{-1}\xi)\hat{\phi}(\xi) \tag{46}$$

where ϕ is the scaling function associated with the scaling factor $S_0(B^{-1}\xi)$,

$$S_l(B^{-1}\xi) = \frac{1}{\sqrt{a}} \sum_{m=0}^{b} S_{lm}(\xi) e^{-i\langle A^{-1}\kappa_m, \xi \rangle} \quad ,$$

and $\{\kappa_0, \ldots, \kappa_b\}$ is a full collectiion of representatives of the coset group $\mathbb{Z}^n / A\mathbb{Z}^n$. Equivalently

$$\psi_l(x) = \sum_{j \in \mathbb{Z}^n} s_{l,j} \phi(Ax - j)$$

where the sequence $\{s_{l,j}\}_{j \in \mathbb{Z}^n}$ is the sequence of coefficients in the expansion

$$S_l(\xi) = \frac{1}{a} \sum_{j \in \mathbb{Z}^n} s_{l,j} e^{-i\langle j, \xi \rangle}$$

By virtue of Proposition 2 the collection $\psi_1, \ldots, \psi_b$ defined by (46) is a full collection of fundamental wavelets.

The only unclear instruction in this procedure involves the selection of elements $\vec{S}_1, \ldots, \vec{S}_b$ in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ so that (45) is a unitary matrix. We clarify this by first considering specific cases.

3.5.1 The case $a = 2$

In this case we don't have much choice. The selection

$$\vec{S}_1(\xi) = (\overline{S_{01}(\xi)}, -\overline{S_{00}(\xi)})$$

does the job nicely. Indeed, every possible choice is of the form

$$\vec{S}_1(\xi) = T(\xi)(\overline{S_{01}(\xi)}, -\overline{S_{00}(\xi)})$$

where $T(\xi)$ is a measurable $2\pi\mathbb{Z}^n$ periodic function which satisfies $|T(\xi)| = 1$ for all ξ.

3.5.2 The case $a = 3$

Once $\vec{S}_1$ is chosen then $\vec{S}_2$ must be of the form

$$\begin{aligned}\vec{S}_2(\xi) &= T(\xi)(\vec{S}_0(\xi) \times \vec{S}_1(\xi)) \\ &= T(\xi)(S_{01}S_{12} - S_{02}S_{11}, S_{02}S_{10} - S_{00}S_{12}, S_{00}S_{11} - S_{01}S_{10})\end{aligned}$$

where $T(\xi)$ is a measurable $2\pi\mathbb{Z}^n$ periodic function which satisfies $|T(\xi)| = 1$ for all ξ and $\times$ denotes the cross product in $\mathbb{C}^3$ which is essentially defined by the second equality.

This is an easy consequence of the fact that any three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{C}^3$ satisfy the identities

$$\langle \vec{w}, \vec{u} \times \vec{v} \rangle = \det \begin{pmatrix} \vec{w} \\ \vec{u} \\ \vec{v} \end{pmatrix} \quad \text{and} \quad |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - |\langle \vec{u}, \vec{v} \rangle|^2 \, .$$

The first implies that $\vec{S}_2$ is orthogonal to both $\vec{S}_0$ and $\vec{S}_1$ while the second implies that $\langle \vec{S}_2(\xi), \vec{S}_2(\xi) \rangle_B = 1$ for all ξ.

The choice of $\vec{S}_1(\xi)$ is not so clear however. On the other hand if the first component of $\vec{S}_0(\xi)$, namely the function $S_{00}(\xi)$, enjoys the property that for some positive constant c

$$|S_{00}(\xi)| \geq c \tag{47}$$

for all ξ then the choice

$$\vec{S}_1 = \frac{(\overline{S_{01}}, -\overline{S_{00}}, 0)}{\sqrt{|S_{00}|^2 + |S_{01}|^2}} \qquad \text{or} \qquad \frac{(\overline{S_{02}}, 0, -\overline{S_{00}},)}{\sqrt{|S_{00}|^2 + |S_{02}|^2}}$$

will do the job. Analogous choices should be clear if $S_{01}(\xi)$ or $S_{02}(\xi)$ satisfy (47). If none of the components of $\vec{S}_0(\xi)$ is uniformly bounded away from 0 then an appropriate combination will to the job. For instance,

$$\vec{S}_1 = T_1(\xi)(\overline{S_{01}}, -\overline{S_{00}}, 0) + T_2(\xi)(\overline{S_{02}}, 0, -\overline{S_{00}}) \, .$$

Note that $T_1(\xi)$ and $T_2(\xi)$ can be chosen so that $\vec{S}_1(\xi)$ is orthogonal to $\vec{S}_0(\xi)$, satisfies the identity $\langle \vec{S}_1(\xi), \vec{S}_1(\xi) \rangle_B = 1$ for all ξ, and is almost as smooth as $\vec{S}_0(\xi)$.

3.5.3 The general case: part 1

For simplicity write

(48) $$\vec{S}_0(\xi) = (u_0(\xi), \ldots, u_b(\xi))$$

and assume that one of its components, which without loss of generality we take to be u_0, satisfies

(49) $$|u_0(\xi)| \geq c > 0$$

for all ξ. Define $\vec{X}_1 = (X_{10}, \ldots, X_{1b}), \ldots, \vec{X}_b = (X_{b0}, \ldots, X_{bb})$ by

(50) $$\begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vdots \\ \vec{X}_b \end{pmatrix} = \begin{pmatrix} -\bar{u}_1 & \bar{u}_0 & 0 & \cdots & 0 \\ -\bar{u}_2 & 0 & \bar{u}_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{u}_b & 0 & 0 & \cdots & \bar{u}_0 \end{pmatrix}$$

and observe that

(51) $$\langle \vec{X}_l(\xi), \vec{S}_0(\xi) \rangle_B = 0$$

for all ξ and all l in $\{1, \ldots, b\}$. For each l in $\{1, \ldots, b\}$ the vector field $\vec{X}_l = (X_{l0}, \ldots, X_{lb})$ may also be described by

$$X_{lm} = \begin{cases} -\bar{u}_l & \text{if } m = 0 \\ \bar{u}_0 & \text{if } m = l \\ 0 & \text{otherwise.} \end{cases}$$

Next note that the Gram matrix

$$\mathbf{Gr}(\vec{X}_1, \ldots, \vec{X}_b) = (\langle \vec{X}_l(\xi), \vec{X}_m(\xi) \rangle_B)_{l,m=1,\ldots,b}$$

can be expressed as

(52) $$\mathbf{Gr}(\vec{X}_1, \ldots, \vec{X}_b) = |u_0|^2 \mathbf{I}_b + |u|^2 \mathbf{P}$$

where $\mathbf{I}_b$ is the $b \times b$ identity matrix,

$$u = (u_1, \ldots, u_b) \qquad \text{and} \qquad |u|^2 = \sum_{j=1}^{b} |u_j|^2 \,.$$

The matrix $\mathbf{P}$ is the orthogonal projection onto the one dimensional subspace generated by

$$u^* = \begin{pmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_b \end{pmatrix}$$

so that in matrix notation

$$|u|^2\mathbf{P} = u^* u \ .$$

Relations (51) and (52) imply that the Gram matrix

$$\mathbf{Gr}(\vec{S}_0, \vec{X}_1, \dots, \vec{X}_b)$$

is uniformly strictly positive definite, in other words it satisfies (43), and that the functions $\psi_1, \dots, \psi_b$ defined by (44) are a full collection of pseudo wavelets.

Orthonormal vector fields $\vec{S}_1, \dots, \vec{S}_b$ which are orthogonal to $\vec{S}_0$ can now be obtained by applying some sort of orthonormalization procedure to the collection $\vec{X}_1, \dots, \vec{X}_b$. For example one may apply the classical Gram Schmidt method to some permutation of $\vec{X}_1, \dots, \vec{X}_b$ to obtain $\vec{S}_1, \dots, \vec{S}_b$.

A more symmetric method of obtaining $\vec{S}_1, \dots, \vec{S}_b$ goes as follows: First write

$$\mathbf{G} = \mathbf{Gr}(\vec{X}_1, \dots, \vec{X}_b) \qquad \text{and} \qquad \mathbf{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vdots \\ \vec{X}_b \end{pmatrix} .$$

Now recall that $\mathbf{G} = \mathbf{X}\mathbf{X}^*$. Since $\mathbf{G}$ is uniformly strictly positive definite we may compute both $\mathbf{G}^{-1}$ and its square root, $\mathbf{G}^{-1/2}$. Let

$$\mathbf{S} = \mathbf{G}^{-1/2}\mathbf{X} \tag{53}$$

where

$$\mathbf{S} = \begin{pmatrix} \vec{S}_1 \\ \vec{S}_2 \\ \vdots \\ \vec{S}_b \end{pmatrix} .$$

Observe that

$$\mathbf{S}\mathbf{S}^* = \mathbf{G}^{-1/2}\mathbf{X}\mathbf{X}^*\mathbf{G}^{-1/2} = \mathbf{G}^{-1/2}\mathbf{G}\mathbf{G}^{-1/2} = \mathbf{I}_b$$

which implies that the rows of $\mathbf{S}$ are orthonormal.

The method for obtaining $\vec{S}_1, \ldots, \vec{S}_b$ which is succinctly summarized by (53) gives rise to explicit expressions for these vector fields because $\mathbf{G}$ can be expressed as

$$\mathbf{G} = (|u_0|^2 + |u|^2)\mathbf{P} + |u_0|^2(\mathbf{I}_b - \mathbf{P}) \,, \tag{54}$$

which is simply a convenient way of writing equation (52). From (54) it follows that

$$\mathbf{G}^{-1/2} = (|u_0|^2 + |u|^2)^{-1/2}\mathbf{P} + |u_0|^{-1}(\mathbf{I}_b - \mathbf{P}) \,,$$

which may be re-written as

$$\mathbf{G}^{-1/2} = \left((|u_0|^2 + |u|^2)^{-1/2} - |u_0|^{-1}\right)\mathbf{P} + |u_0|^{-1}\mathbf{I}_b$$

or, which is even more convenient, as

$$\mathbf{G}^{-1/2} = \left(\frac{1}{\sqrt{|u_0|^2 + |u|^2}} - \frac{1}{|u_0|}\right)\frac{1}{|u|^2}u^*u + \frac{1}{|u_0|}\mathbf{I}_b \,. \tag{55}$$

Formula (55), the expression for $\mathbf{X}$ in terms of u and $\mathbf{I}_b$, and the fact that $|u_0|^2 + |u|^2 = 1$ leads to

$$\mathbf{G}^{-1/2}\mathbf{X} = \begin{pmatrix} -\bar{u}_1 & \beta\bar{u}_1u_1 + \alpha & \beta\bar{u}_1u_2 & \cdots & \beta\bar{u}_1u_b \\ -\bar{u}_2 & \beta\bar{u}_2u_1 & \beta\bar{u}_2u_2 + \alpha & \cdots & \beta\bar{u}_2u_b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{u}_b & \beta\bar{u}_bu_1 & \beta\bar{u}_bu_2 & \cdots & \beta\bar{u}_bu_b + \alpha \end{pmatrix}$$

where

$$\alpha = \frac{\bar{u}_0}{|u_0|}$$

and

$$\beta = \frac{-\alpha}{1 + |u_0|} \,.$$

In other words, in this case for each l in $\{1,\ldots,b\}$ the vector field $\vec{S}_l = (S_{l0},\ldots,S_{lb})$ may be described by

$$S_{lm} = \begin{cases} -\bar{u}_l & \text{if } m=0 \\ \beta\bar{u}_l u_l + \alpha & \text{if } m=l \\ \beta\bar{u}_l u_m & \text{otherwise.} \end{cases} \tag{56}$$

We summarize this discussion as follows:

Proposition 3 *If one component of $\vec{S}_0$ is uniformly bounded away from 0 then the above procedures give rise to explicit formulas for mutually orthonormal vector fields $\vec{S}_1,\ldots,\vec{S}_b$ in $VF(R^n/2\pi\mathbb{Z}^n,\mathbb{C}^a)$ which are orthogonal to $\vec{S}_0$. The corresponding functions $\psi_1,\ldots,\psi_b$ defined by (44) are a full collection of orthonormal wavelets.*

In particular, if $\vec{S}_0(\xi) = (u_0(\xi),\ldots,u_b(\xi))$ and $|u_0(\xi)| \geq c > 0$ then the vector fields $\vec{S}_1,\ldots,\vec{S}_b$ whose components are described by (56) are mutually orthonormal and are orthogonal to $\vec{S}_0$. The corresponding functions $\psi_1,\ldots,\psi_b$ defined by (44) are a full collection of orthonormal wavelets.

Remark There are many interesting cases where the hypothesis of the above proposition is valid. For instance, suppose the scaling factor $S(B^{-1}\xi)$ is real valued and non-negative; examples of this will be given in Subsection 3.7. The first component u_0 of $\vec{S}_0(\xi)$ is given by

$$u_0 = \frac{1}{\sqrt{a}} \sum_{\nu\in\mathcal{K}_B} S(B^{-1}(\xi - 2\pi\nu))\,.$$

Since

$$1 = \sum_{\nu\in\mathcal{K}_B} |S(B^{-1}(\xi-2\pi\nu))|^2 \leq \sum_{\nu\in\mathcal{K}_B} S(B^{-1}(\xi-2\pi\nu)) \leq a$$

it follows that

$$\frac{1}{\sqrt{a}} \leq u_0 \leq \sqrt{a}\,.$$

Details Note that in matrix notation $\mathbf{X}$ can be expressed as

$$\mathbf{X} = [u^*, \bar{u}_0\mathbf{I}_b]$$

and hence

$$\mathbf{G} = \mathbf{XX}^* = [u^*, \bar{u}_0\mathbf{I}_b] \begin{bmatrix} -u \\ u_0\mathbf{I}_b \end{bmatrix}$$
$$= u^*u + |u_0|^2\mathbf{I}_b \,.$$

Thus

$$\mathbf{G}^{-1/2} = \left(\frac{1}{\sqrt{|u_0|^2 + |u|^2}} - \frac{1}{|u_0|} \right) \frac{1}{|u|^2} u^*u + \frac{1}{|u_0|}\mathbf{I}_b$$

and, since $|u_0|^2 + |u|^2 = 1$, we may write

$$\mathbf{G}^{-1/2}\mathbf{X} = \left[\left(1 - \frac{1}{|u_0|}\right) \frac{1}{|u|^2} u^*u + \frac{1}{|u_0|}\mathbf{I}_b \right] [-u^*, \bar{u}_0\mathbf{I}_b]$$
$$= \left[-u^*, \frac{\bar{u}_0}{|u_0|} \left(\frac{|u_0| - 1}{|u|^2} u^*u + \mathbf{I}_b \right) \right] .$$

Using the substitution $|u|^2 = 1 - |u_0|^2 = (1 - |u_0|)(1 + |u_0|)$ in the last expression results in (56).

3.5.4 The general case: part 2

Finally suppose no components of $\vec{S}_0(\xi) = (u_1(\xi), \ldots, u_b(\xi))$ are uniformly bounded away from zero. Since $\langle \vec{S}_0, \vec{S}_0 \rangle_B = 1$ it follows that there are measurable sets $\Omega_0, \ldots, \Omega_b$ which enjoy the following properties

- Each set is $2\pi\mathbb{Z}^n$ periodic. In other words, if ξ is in Ω_m then $\xi + 2\pi k$ is in Ω_m for all k in $\mathbb{Z}^n$ and this hold for all m in $\{0, \ldots, b\}$.
- Whenever ξ is in Ω_m then

$$|u_m(\xi)| \geq \frac{1}{2\sqrt{a}} \,. \tag{57}$$

- They are mutually disjoint. That is, $\Omega_l \bigcap \Omega_m = \emptyset$.
- Their union covers $\mathbb{R}^n$. That is,

$$\bigcup_{m \in \{0, \ldots, b\}} \Omega_m = \mathbb{R}^n \,. \tag{58}$$

Let $\chi_0(\xi), \ldots, \chi_b(\xi)$ be the characteristic functions of of $\Omega_0, \ldots, \Omega_b$ respectively. The last two listed properties of the Ω's imply that

$$\sum_{m=0}^{b} \chi_m(\xi) = 1$$

for all ξ. In view of the property described by equation (57) using the methods outlined above for each m in $\{0, \ldots, b\}$ we can construct vector fields $\vec{S}_1^m, \ldots, \vec{S}_b^m$ in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ such that

$$\langle \vec{S}_k^m, \vec{S}_0 \rangle_B = 0 \qquad \text{for all } k \in \{1, \ldots, b\}$$

and

$$\langle \vec{S}_k^m, \vec{S}_l^m \rangle_B = \delta_{k,l}\chi_m(\xi) \qquad \text{for all } k, l \in \{1, \ldots, b\} .$$

These properites imply that the vector fields $\vec{S}_1, \ldots, \vec{S}_b$ defined by

$$\vec{S}_l = \sum_{m \in \{0, \ldots, b\}} \vec{S}_l^m \tag{59}$$

are mutually orthonormal vector fields in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ which are orthogonal to $\vec{S}_0$. We summarize this last discussion as follows:

Proposition 4 *The procedure outlined above gives rise to mutually orthonormal vector fields $\vec{S}_1, \ldots, \vec{S}_b$ in $VF(R^n/2\pi\mathbb{Z}^n, \mathbb{C}^a)$ which are orthogonal to $\vec{S}_0$. The corresponding functions $\psi_1, \ldots, \psi_b$ defined by (44) are a full collection of orthonormal wavelets.*

3.6 Remarks

Note that in the case $a = 2$ there is in some sense a "canonical" method of obtaining S_1 from S_0. This is essentially the formula given by in Subsection 3.5.1. Unfortunately, as is quite evident from the construction outlined above, no such canonical method is available in the case $a > 2$.

3.6.1 Cases which simplify

In certain cases the general procedure outlined above is not only unnecessarily cumbersome but may not produce wavelets with desired properties.

Sometimes specialized methods may be simpler and produce better results. A good example of this is illustrated by the method of obtaining the wavelets in Subsection 3.7.1.

Other examples arise if we recall that the second step of the recipe is equivalent to finding functions $S_1(\xi), \ldots, S_b(\xi)$ which are $2\pi\mathbb{Z}^n$ periodic and such that

$$(60) \quad \begin{pmatrix} S_0(\xi - 2\pi B^{-1}\nu_0) & S_0(\xi - 2\pi B^{-1}\nu_1) & \cdots & S_0(\xi - 2\pi B^{-1}\nu_b) \\ S_1(\xi - 2\pi B^{-1}\nu_0) & S_1(\xi - 2\pi B^{-1}\nu_1) & \cdots & S_1(\xi - 2\pi B^{-1}\nu_b) \\ \vdots & \vdots & \ddots & \vdots \\ S_b(\xi - 2\pi B^{-1}\nu_0) & S_b(\xi - 2\pi B^{-1}\nu_1) & \cdots & S_b(\xi - 2\pi B^{-1}\nu_b) \end{pmatrix}$$

is an orthogonal matrix for all ξ where $\mathcal{K}_B = \{\nu_0, \ldots, \nu_b\}$ is a full collection of representatives of $\mathbb{Z}^n/B\mathbb{Z}^n$. In certain cases it is possible to do this directly.

For instance if $a = 2$, or equivalently $b = 1$, then the choice

$$(61) \qquad S_1(\xi) = \overline{S_0(\xi - 2\pi B^{-1}\nu_1)}e^{-i\langle\mu_1, \xi - k\rangle}$$

does the job, where mu_1 is the non-zero element in $\mathcal{K}_A$ and k is any any conveniently chosen element of $\mathbb{Z}^n$. This is equivalent to the choice of S_1 given in Subsection 3.5.1. Formula (61) is essentially the classical formula used to construct univariate wavelets in the dyadic case, see [14, 31, 33].

Another such example arises in the case $n = 2$, $A = B = 2I$ when $S_0(\xi)$ is real valued. If

$$\nu_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \nu_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \nu_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then one can verify that

$$S_1(\xi) = S_0(\xi - \pi\nu_1)e^{-i\langle\nu_2, \xi\rangle},$$

$$S_2(\xi) = S_0(\xi - \pi\nu_2)e^{-i\langle\nu_3, \xi\rangle},$$

and

$$S_3(\xi) = S_0(\xi - \pi\nu_3)e^{-i\langle\nu_1, \xi\rangle}$$

do the the job, see [20, 33]. More generally, if $a = 4$ and $S_0(\xi)$ is real valued an analogous choice of S_1, S_2, and S_3 is possible whenever $\mathbb{Z}^n/B\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^2/2\mathbb{Z}^2$. We remind the reader that in the case $a = 4$ the group $\mathbb{Z}^n/B\mathbb{Z}^n$ can be isomorphic to one of two abelian groups, $\mathbb{Z}^2/2\mathbb{Z}^2$ or $\mathbb{Z}/4\mathbb{Z}$.

3.6.2 Decay properties

A desirable feature for the wavelets $\psi_1, \ldots, \psi_b$ is that they enjoy the same decay properties as the scaling function which they are associated with. For example if the scaling function ϕ has compact support or exponential decay then it would be nice if each of the ψ_j's enjoyed the same property. Unfortunately the general recipe outlined above does not necessarily give rise to such wavelets. The reason for this is that, roughly speaking, if $a > 2$ the functions $S_1, \ldots, S_b$ produced by this method do not in general have the same smoothness properties as S_0.

There are many specific cases where one can directly construct wavelets which enjoy this feature. For example, it is not difficult to find compactly supported wavelets in that case where the scaling function is the characteristic function of a compact set, see Subsection 3.7.1. On the other hand a comprehensive treatment of this issue is beyond the scope of this presentation. Here we mention several fairly wide cases.

The case $a = 2$ This case is well understood and presents no difficulties. The choice for S_1 given by (61) gives rise to a wavelet $\psi = \psi_1$ with essentially the same decay properties as the corresponding scaling function ϕ. For example, if ϕ has compact support then the scaling factor S_0 must be a polynomial, in other words the scaling sequence must be finite. In this case the function S_1 given by (61) is also a polynomial which implies that the corresponding wavelet has compact support.

The case $a = 3$ The general recipe outlined in Subsection 3.5.4 does not in general give rise to S_1 and S_2 which are as smooth as S_0. On the other hand the procedure indicated in Subsection 3.5.2 gives rise to S_1 and S_2 which have the following properties:

- If S_0 is a polynomial or is analytic then S_1 and S_2 may fail to have this property. However both S_1 and S_2 will be infinitely differentiable.
- If S_0 is k times continously differentiable then both S_1 and S_2 enjoy this property.

The case $a > n$ The abstract method used by K. Gröchenig [18], see also [33], in the case $A = 2I$ can be adapted in this case to prove the existence of $S_1, \ldots, S_b$ which have the following properties:

- If S_0 is a polynomial the functions $S_1, \ldots, S_b$ may fail to be polynomials. However, they will be analytic.
- If S_0 is analytic or k times continously differentiable then functions $S_1, \ldots, S_b$ enjoy the same property.

A constructive variant of this method has been derived by P. Vial.

Compact support The desirability of nice decay properties, particulary compact support, has lead in the univariate dyadic case to the study of "biorthogonal" scaling functions and wavelets, see [7, 42], and in the mutivariate dyadic case to a detailed study of various pseudo or pre scaling functions and wavelets associated with box splines, see for example [22, 25, 35, 38].

3.7 Examples

3.7.1 Wavelets generated by self similar sets

Consider the multiresolution analyses discussed in Subsection 2.5.1. Namely, suppose $\mathcal{K}_A = \{\kappa_0, \ldots, \kappa_b\}$ is a full collection of representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$ and Q is the compact set defined by (16) which is

$$Q = \{x \in \mathbb{R}^n : x = \sum_{j=1}^{\infty} A^{-j}\epsilon_j,\ \epsilon_j \in \mathcal{K}_A\}\ .$$

Furthermore suppose $|Q| = 1$. Then χ_Q is a scaling function for the multiresolution analysis it generates. In this case a fundamental set of wavelets can be constructed directly as follows:

Recall that $|\det A| = a$ and $b = a - 1$. Select an $a \times a$ orthogonal matrix of the form

$$\frac{1}{\sqrt{a}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_{1,0} & u_{1,1} & \cdots & u_{1,b} \\ \vdots & \vdots & \ddots & \vdots \\ u_{b,0} & u_{b,1} & \cdots & u_{b,b} \end{pmatrix}$$

and for $j = 1, \ldots, b$ set

$$\psi_j(x) = \sum_{m=0}^{b} u_{j,m} \chi_Q(Ax - \kappa_m) \,. \tag{62}$$

The functions $\psi_1, \ldots, \psi_b$ are a full collection of fundamental wavelets. The corresponding orthonormal wavelet basis for $L^2(I\!R^n)$

$$\{a^{m/2}\psi_j(A^m x - k)\}_{j\in\{1,\ldots,b\},\, m\in\mathbf{Z},\, k\in\mathbf{Z}^n}$$

is a natural generalization of the classical Haar basis for $L^2(I\!R)$.

Univariate examples If $A = 2$ and $\mathcal{K}_A = \{0, 1\}$ then Q is the interval $[0, 1]$ and $\psi = \psi_1$ is the function in Subsection 1.2.1. The collection

$$\{2^{m/2}\psi(2^m x - k)\}_{k,m\in\mathbf{Z}}$$

is the classical Haar basis for $L^2(I\!R)$.

More generally if $A = N$, $N \geq 2$, and $\mathcal{K}_A = \{0, \ldots, N-1\}$ then Q is the interval $[0, 1]$, $b = N - 1$ and the wavelets $\psi_1, \ldots, \psi_b$ are the functions described in Subsection 1.4.1. All the cases when Q is an interval give rise to similar fundamental wavelets which are step functions. On the other hand if Q is not an interval then the corresponding wavelets are simple functions which are somewhat cumbersome to portray, the bivariate analogues are easier to describe via approprate plots.

Bivariate examples If $A = 2I$ and the members of $\mathcal{K}_A = \{\kappa_0, \ldots, \kappa_3\}$ are described by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

then Q is the unit square $[0, 1]^2$. Of course there are many possible choices for the coefficients in represntation (62). The popular choice

$$\begin{pmatrix} u_{1,0} & u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,0} & u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,0} & u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

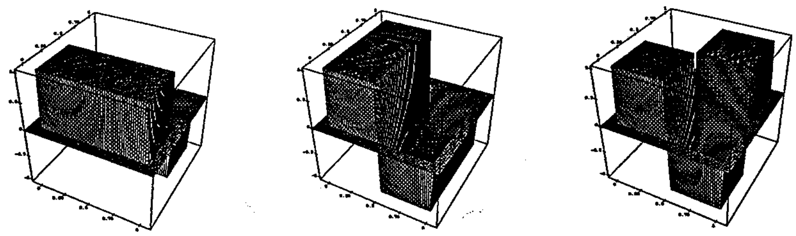

Figure 7: The Haar like wavelets ψ_1, ψ_2 and ψ_3.

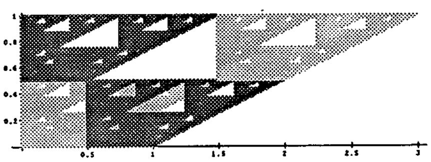

Figure 8: A representation of the wavelet ψ_3 associated to the tile corresponding to $A = 2I$ and $\mathcal{K}_A$ described by the columns of ??

gives rise to a bivariate analogue of the Haar system with the fundamental wavelets ψ_1, ψ_2, and ψ_3 plotted in Figure 7.

On the other hand if the members of $\mathcal{K}_A$ are described by the columns of

$$\begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

then corresponding tile Q together with the coefficients $u_{i,j}$ given above also gives rise to three wavelets ψ_1, ψ_2, and ψ_3. The gray level plot of ψ_3 is displayed in Figure 8.

If the matrix A and the set of digits $\mathcal{K}_A$ are the same as those which give rise to the so-called twin dragon set Q then corresponding fundamental wavelet has values 1 and -1. Its gray level plot is displayed in Figure 9.

We finish our discussion of this class of examples by considering the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

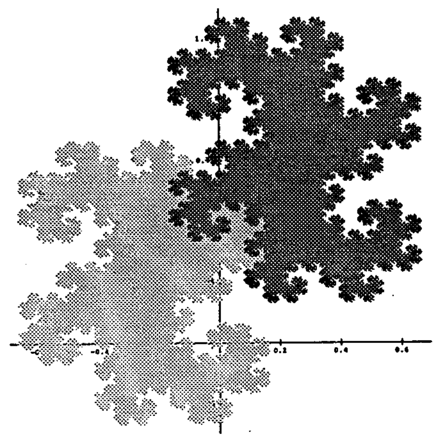

Figure 9: A represntation of the wavelet associated with the twin dragon tile.

Figure 10: A representation of the wavelets in the last tile example.

which has determinant 3. If the members of $\mathcal{K}_A$ are described by the columns of

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

then corresponding tile Q together with the coefficients $u_{i,j}$ given by

$$\begin{pmatrix} u_{1,0} & u_{1,1} & u_{1,2} \\ u_{2,0} & u_{2,1} & u_{2,2} \end{pmatrix} = \sqrt{3} \begin{pmatrix} \frac{1}{2} & -1 & \frac{-1}{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

give rise to fundamental wavelets ψ_1 and ψ_2 whose gray level plots are displayed in Figure 10.

3.7.2 Wavelets generated by univariate polynomial splines

For any positive integer k consider the dyadic multiresolution analysis of $L^2(\mathbb{R})$ which has the function ϕ defined by

$$\hat{\phi}(\xi) = \frac{\xi^{-k}}{\sqrt{\sum_{l\in\mathbf{Z}} |\xi + 2\pi l|^{-2k}}}$$

as a scaling function. See Subsection 2.5.2. The corresponding scaling factor is given by

$$S(\xi) = \left\{ \frac{\sum_{m\in\mathbf{Z}} |\xi + 4\pi m|^{-2k}}{\sum_{m\in\mathbf{Z}} |\xi + 2\pi m|^{-2k}} \right\}^{1/2} = \frac{1}{2} \sum_{m\in\mathbf{Z}} s_k e^{-im\xi}$$

where the coefficients $\{s_m\}$ are real because S is real and even. Hence the wavelet $\psi = \psi_1$ resulting from the recipe, see (61), can be expressed as

$$\hat{\psi}(\xi) = e^{-i\xi/2}\overline{S((\xi/2) - \pi)}\hat{\phi}(\xi) \tag{63}$$

or

$$\psi(x) = \sum_{m\in\mathbf{Z}} (-1)^{1-m} s_{1-m} \psi(2\xi - m) \, .$$

Note that for any positive integer k the sum

$$\sum_{m\in\mathbf{Z}} |\xi + 2\pi m|^{-2k}$$

can expressed in terms of elementary trigonometric functions. For $k = 1$ this is the well known identity

$$\sum_{m\in\mathbf{Z}} |\xi + 2\pi m|^{-2} = \frac{4}{\sin^2 \xi/2}$$

and thus for $k = 2, 3, \ldots$,

$$\sum_{m\in\mathbf{Z}} |\xi + 2\pi m|^{-2k} = \frac{1}{(2k-1)!} \frac{d^{2k-2}}{d\xi^{2k-2}} \frac{4}{\sin^2 \xi/2} \, .$$

Thus the Fourier transforms, (63), of these fundamental wavelets can be expressed in terms of elementary functions. Other formulas related to this example can be found in [5, 14, 33].

Before leaving this section we mention that, like the classical Haar system, these wavelets were discovered without the framework provided by multiresolution analysis. They were discovered by J. Stromberg [40] shortly before this framework was introduced.

3.7.3 Multivariate analogues of the univariate spline examples

Here we indicate some of the details only for the polyharmonic splines. Various details concerning wavelets arising from the multiresolution analyses generated by convolutional iterates of characteristic functions can be found in [39]. For detailed constructions of wavelets and psuedo or pre wavelets generated by the box spline multiresolution analyses see [22, 25, 35, 38].

Polyharmonic splines For any integer k greater that $n/4$ consider the multiresolution analysis of $L^2(I\!R^n)$ associated to $(Z\!\!Z^n, A)$ which has the function ϕ defined by

$$\hat{\phi}(\xi) = \frac{|\xi|^{-2k}}{\sqrt{\sum_{l \in \mathbb{Z}^n} |\xi + 2\pi l|^{-4k}}}$$

as a scaling function. Assume that A is a similarity and an acceptable dilation for $Z\!\!Z^n$. See Subsection 2.5.3. The corresponding scaling factor is

$$S(\xi) = \left\{ \frac{\sum_{m \in \mathbb{Z}} |\xi + 2\pi Bm|^{-4k}}{\sum_{m \in \mathbb{Z}} |\xi + 2\pi m|^{-4k}} \right\}^{1/2} \tag{64}$$

where $B = A^*$.

In view of the remark recorded immediately after the statement of Proposition 3 the fact that scaling factor (64) is real valued and non-negative implies that the method outlined in Subsection 3.5.3 can be used to to obtain explicit formulas for the Fourier transforms of a collection of fundamental wavelets. Like the scaling function ϕ these wavelets enjoy exponential decay as $|x|$ goes to infinity.

Note that in the case $a = 2$ the formula for the wavelet $\psi = \psi_1$ is analogous to formula given in the univariate case.

3.7.4 Compactly supported wavelets

To see compactly supported wavelets which are significantly smoother than the examples found in Subsection 3.7.1 consider the univariate examples corresponding to the scaling functions mentioned in Subsection 2.5.4. The simplest specific example are the functions ϕ and ψ described in Subsubsection 1.2.3 and plotted in Figure 2. Since $a = 2$ in these cases, an application of

(61) gives rise to a wavelet with the desired properties; that is, if

$$\phi(x) = \sum_{j=0}^{2m+1} s_j \phi(2x - j)$$

then

$$\psi(x) = \sum_{j=0}^{2m+1} (-1)^j s_{2m+1-j} \psi(2x - j)$$

where the k in formula (61) is chosen so that ψ has support in the same interval as ϕ.

As mentioned in Subsection 2.5.4 multivariate examples can be constructed by taking tensor products of univariate examples. For instance, if ϕ_i and ψ_i are scaling function and wavelet associated to a univariate dyadic multiresolution anslysis $\mathcal{V}_i$, $i = 1, 2$, then

$$\varphi_0(x, y) = \phi_1(x)\phi_2(y)$$

is a scaling function for a bivariate dyadic multiresolution analysis $\mathcal{V}$ and

$$\varphi_1(x, y) = \phi_1(x)\psi_2(y)$$

$$\varphi_2(x, y) = \psi_1(x)\phi_2(y)$$

$$\varphi_3(x, y) = \psi_1(x)\psi_2(y)$$

are a full collection of fundamental wavelets associated with $\mathcal{V}$.

As indicated in Subsection 2.5.4 a theory of smooth compactly supported wavelets associated with a general dilation matrix A has not yet been developed. We close this section with the following amusing example:

Consider the acceptable dilation of $\mathbb{Z}^2$

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

and the bivariate two scale difference equation

$$\varphi_0(x) = \sum_{j=0}^{3} s_j \varphi_0(Ax - k_j)$$

where $\{s_0, s_1, s_2, s_3\}$ is the same as that listed in Subsection 1.2.3 and $\{k_0, k_1, k_2, k_3\}$ are the columns of

$$\begin{pmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The compactly supported solution φ_0 of this difference equation is a scaling function for a multiresolution analysis associated to $(\mathbb{Z}^2, A)$. Both φ_0 and an associated fundamental wavelet

$$\varphi_1(x) = \sum_{j=0}^{3}(-1)^j s_{3-j}\varphi_1(Ax - k_j)$$

can be expressed in terms of the univariate scaling function and wavelet, ϕ and ψ, decribed in Subsection 1.2.3, namely

$$\varphi_0(x) = \phi(x_1)\phi(x_2)$$

and

$$\varphi_1(x) = \psi(x_1)\phi(x_2)$$

where x_1 and x_2 are the first and second conponents of x respectively.

3.7.5 Band limited wavelets

Let Ω be a compact subset of $\mathbb{R}^n$ which has the four properties listed in the beginning of Subsection 2.5.5 and let $\mathcal{V}$ be the multiresolution analysis associated to $(\mathbb{Z}^n, A)$ which has ϕ, where $\hat{\phi}(\xi) = \chi_\Omega(\xi)$, as a scaling function. Suppose $\Omega_1, \ldots, \Omega_b$ are mutually essentially disjoint subsets of $B\Omega$ each of which which are congruent to Ω mod $2\pi\mathbb{Z}^n$ and essentially disjoint from Ω. In other words, if $\Omega_0 = \Omega$ then

- $\Omega_i \subset B\Omega$ for each $i \in \{0, 1, \ldots, b\}$.
- $|\Omega_i \bigcap \{\Omega_i + 2\pi k\}| = 0$ for any element k in $\mathbb{Z}^n \setminus \{0\}$ and each $i \in \{0, 1, \ldots, b\}$.
- For each $i \in \{0, 1, \ldots, b\}$

$$\bigcup_{k \in Z^n} \{\Omega_i + 2\pi k\} = \mathbb{R}^n.$$

Finally, for each $i \in \{1, \dots, b\}$ consider the function ψ defined by

$$\hat{\psi}_i(\xi) = \chi_{\Omega_i}(\xi) \,.$$

Then the functions $\psi_1, \dots, \psi_b$ are a collection of fundamental wavelets associated with $\mathcal{V}$.

Just as the scaling function ϕ these wavelets have poor decay properties at infinity.

The univariate case where $A = B = 2$ and $\phi(x) = \operatorname{sinc} x$ the corresponding wavelet ψ_1 is the sinc wavelet ψ mentioned in Subsection 1.2.2 and plotted in Figure 1.

In the bivariate case where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and $\phi(x) = \operatorname{sinc} x$ then the corresponding wavelet ψ_1 is the sinc wavelet ψ mentioned in Subsection 1.4.2 and plotted in Figure 3.

Wavelets with better decay properties These, of course, are derived from the scaling functions with better decay properties mentioned in Subsection 2.5.5. Unfortunately their construction is not as simple as that found above. In the case $a = b + 1 = 2$ we can apply the canonical procedure which results from the use of formula (61). Fortunely, since the scaling factor is non-negative, in the case $a > 2$ the procedure outlined in Subsection 3.5.3 can be applied to obtain the S_i's to get the corresponding ψ_i's. In either case the wavelets will enjoy the same decay properties as the scaling function.

References

[1] P. Auscher and P. Tchamitchian, Bases d'ondelettes sur les courbes corde-arc, noyau de Cauchy et esoaces de Hardy associés, *Rev. Mat. Iberoamericana* 5 (1989), 139-170.

[2] M. A. Berger and Y. Wang, Multidimensional two-sacle dilation equations, in *Wavelets: a tutorial in Theory and Applications*, C. K. Chui , ed., Academic Press, Boston, 1992, 295-323.

[3] G. Beylkin, R. Coifman, and V. Rokhlin, Fast wavelet transforms and numerical algorithms I, *Com. Pure and App. Math.*, 44 (1991), 141-183.

[4] C. Chui, *An Introduction to Wavelets*, Academic Press, Boston, 1992.

[5] C. Chui, *Wavelets: a Tutorial in Theory and Practice*, Academic Press, Boston, 1992.

[6] A. Cohen, Ondelettes, analyses multirésolutions et filtres miroirs en quadrature, *Ann. Inst. Henri Poincaré* 7 (1990), 439-459.

[7] A. Cohen, I. Daubechies, and J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, AT&T Technical Report, 1990.

[8] A. Cohen, and I. Daubechies, Nonseparable Bidimensional Wavelet Bases, Preprint, AT&T Bell Laboratories, New Jersey, 1991, pp. 1-94.

[9] A. Cohen, and I. Daubechies, Orthonormal Bases of Compactly Supported Wavelets, III. Better Frequency Resolution, preprint.

[10] D. Colella, and C. Heil, Characterizations of Scaling Functions, I. Continuous Solutions, MITRE Corporations, McLean, Virginia, 1991.

[11] J. M. Combes, A. Grossmann, and P. Tchamitchian, Wavelets, Time-Frequency Methods and Phase Space, *Lecture Notes on IPTI*, Springer-Verlag, Berlin, New York, 1989.

[12] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. in Pure and Applied Math.* 41 (1988), 909-996.

[13] I. Daubechies and J. C. Legarias, Two-scale difference operations 1. Existence and global regularity of solutions, *SIAM J. Math. Anal.* 22, (1991), 1388-1410.

[14] I. Daubechies, *Ten Lectures on Wavelets*, Vol. 61 in the CBMS-NSF Regional Concerence Series in Applied Mathematics, SIAM, Philadelphia, 1992.

[15] H. G. Feichtinger and K. Gröchenig, Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretic point of view, in *Wavelets: a tutorial in Theory and Applications*, C. K. Chui , ed., Academic Press, Boston, 1992, 359-397.

[16] M. Frazier, B. Jaweth, and G. Wiess, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS-CLN 79, American Mathematical Society, Providence, 1991.

[17] K. Gröchenig and W.R. Madych, Multiresolution analyses, Haar bases, and self-similar tilings of R^n, BRC technical report, to appear in *Trans. IEEE.*

[18] K. Gröchenig, Analyse multi-echelle et bases d'ondelettes, *Comp. Rend. Acad. Sci. Paris*, 305, Serie I, (1987),13-17.

[19] C. E. Heil and D. F. Walnut, Continuous and Discrete Wavelet Transforms, SIAM Review 31-4 (1989), 628-666.

[20] S. Jaffard, P. G. Lemarie, S. Mallat, and Y. Meyer, Dyadic multiscale analysis of $L^2(R^n)$, manuscript.

[21] S. Jaffard, Wavelet methods for fast resolution of elliptic problems, LAMM Technical report.

[22] R.-Q. Jia and C. A. Micchelli, Using the refinement equations for the construction of pre-wavelets II: Powers of two, *Curves and Surfaces*, P.-J. Laurent, A. Le Mehaute, L. Schumaker, eds., Academic Press, San Diego, 1991.

[23] P. G. Lemarié, Theorie L^2 des surfaces-splines, preprint 1987.

[24] P. G. Lemarié, Bases d'Ondelettes sur les Groupes de Lie Stratifiés, *Bull. Soc. Math. France,* to appear.

[25] R. A. H. Lorentz and W.R. Madych, Wavelets and Generalized Box Splines, GMD technical report, to appear in *Applicable Analysis.*

[26] R. A. H. Lorentz and W.R. Madych, Spline wavelets for ordinary differential equations, GMD technical report.

[27] W. R. Madych, Cardinal interpolation with polyharmonic splines, *Multivariate Approximation IV*, C. Chui, W. Schepp and K. Zeller, eds., Int. Ser. Num. Math. Vol. 90, Birkhäuser Verlag, Basel, 1989, 241-248.

[28] W. R. Madych, Polyharmonic splines, multiscale analysis, and entire functions, in *Multivariate Approximation and Interpolation,* W. Haußmann and K. Jetter, eds., Birkhäuser Verlag, Basel, 1990, 205-216.

[29] W. R. Madych, Multiresolution analysis, wavelets, and homogeneous functions, BRC preprint.

[30] W. R. Madych, Some elementary properties of multiresolution analyses of $L^2(I\!R^n)$, in *Wavelets: a tutorial in Theory and Applications*, C. K. Chui , ed., Academic Press, Boston, 1992, 259-294.

[31] S. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$, *Trans. Amer. Math. Soc.* 315 (1989), 69-88.

[32] Y. Meyer, Ondelettes, functions splines et analyses graduées. Rapport CEREMADE n.8703, 1987.

[33] Y. Meyer, *Ondelettes et Opérateurs*, Hermann, Paris, 1990.

[34] S. D. Riemenschneider, Multivariate Cardinal Interpolation, *Approximation Theory VI*, C. K. Chui, L. L. Schumaker, and J. G. Ward, eds., Academic Press, New York, 1989, 561-584.

[35] S. D. Riemenschneider and Z. W. Shen, Wavelets and pre-wavelets in low dimensions, Univ. Alberta, preprint.

[36] W. Rudin, *Fourier Analysis on Groups*, Intersience, Wiley, New York, 1962.

[37] I. J. Schoenberg, *Cardinal Spline Interpolation*, CBMS Vol. 12, SIAM, Philadelphia, 1973.

[38] J. Stöckler, Multivariat wavelets, in *Wavelets: a tutorial in Theory and Applications*, C. K. Chui , ed., Academic Press, Boston, 1992, 325-355.

[39] R. S. Strichartz, Wavelets and Self-Affine Tilings, preprint.

[40] J. O. Stromberg, A modified Franklin system and higher order spline systems on R^n as unconditional bases for Hardy spaces, Proceeding of a conference in honor of Antoni Zygmund, Vol. II, W. Beckner et al, eds., Wadsworth, N.Y., 1981, 475-493.

[41] P. Tchamitchian, Ondelettes et Intégrale de Cauchy sur les Courbes Lipschitziennes, *Annals of Math.* 129 (1989), 641-649.

[42] Biorthogonalité et théorie des opérateurs, *Rev. Math. Iberoamer.*, 3 (1987), 163-189.

[43] B. Torrésani, Wavelet Analysis of Asymptotic Signals, Centre de Physique Théorique, Marseille, preprint.

[44] M. Vetterli, Multirate Filter Banks for Subband Coding, Technical Report CU/CTR/TR-165-90-02, Center for Telecommunication Research, Columbia University, New York, 1989, 1-57.

Scaling Functions and Sequences Associated with Orthonormal Wavelets

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Abstract

It is well known that the integer translates of the scaling function associated with a given scaling sequence may fail to be mutually orthogonal. Here, we address several technical questions related to this phenomenon. For instance, we show that the scaling function naturally associated with a finite scaling sequence always generates a multiresolution analysis and give elementary but non-trivial examples of scaling sequences which give rise to pathological scaling functions.

1 Introduction

Suppose φ satisfies the two scale difference equation

$$\varphi(x) = \sum_k s_k \varphi(2x - k), \tag{1}$$

where x is any real number and $\{s_k\}$ is a finite sequence of scalars. Then a necessary condition that $\{\varphi(x-k)\}_{k\in\mathbf{Z}}$ be an orthonormal system of functions, namely

$$\int_{-\infty}^{\infty} \varphi(x-m)\overline{\varphi(x-n)}dx = \delta_{m,n} \ , \tag{2}$$

which generates a dyadic multiresolution analysis is that the sequence $\{s_k\}$ satisfies the following constraints:

$$\sum_k s_k \bar{s}_{k-2n} = 2\delta_{0,n} \tag{3}$$

and

$$\sum_k s_k = 2, \tag{4}$$

where $\delta_{m,n}$ is the Kronecker delta. The sequence $\{s_k\}$ is often referred to as a *scaling sequence* and, if both (1) and (2) hold, the function φ is called a *scaling function.* Relation (3) is a consequence of (1) and (2) while relation (4) follows from

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1,$$

a condition which is implied whenever the closed linear span of $\{\varphi(x-k)\}_{k\in \mathbf{Z}}$ generates a dyadic multiresolution analysis. See [4] for more details.

1.1

Recall that if $\{s_k\}$ is a finite scaling sequence which satisfies (3) and (4) the canonical solution of (1) is given by the function φ which is well defined by the formula for its Fourier transform

$$\hat{\varphi}(\xi) = \prod_{k=1}^{\infty} S(\xi/2^k), \tag{5}$$

where

$$S(\xi) = \frac{1}{2}\sum_k s_k e^{-ik\xi} \tag{6}$$

and

$$\hat{\varphi}(\xi) = \int_{-\infty}^{\infty} \varphi(x)e^{-i\xi x}dx. \tag{7}$$

Unfortunately, examples show that φ need not satisfy (2). See Subsection 1.4.

On the other hand such a φ always generates a dyadic multiresolution analysis. To be more precise let $V(\varphi)$ be the closed linear span of $\{\varphi(x-k)\}_{k\in \mathbf{Z}}$ in $L^2(\mathbb{R})$. We say that φ generates a multiresolution analysis whenever there is a multiresolution analysis

$$\mathcal{V} = \{... \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...\}$$

with $V_0 = V(\varphi)$.

Theorem 1 *Suppose $\{s_k\}$ is a finite scaling sequence which satisfies (3) and (4) and φ is the function which is described by (5). Then φ generates a dyadic multiresolution analysis.*

If φ fails to satisfy (2) then it is somewhat technical and awkward to describe a scaling function and corresponding scaling coefficients for the multiresolution analysis which is generated by φ. The Fourier transform of such a scaling function φ_0 is essentially given by

$$\hat{\varphi}_0(\xi) = \frac{\hat{\varphi}(\xi)}{\sqrt{\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi + 2\pi k)|^2}}.$$

If all the coefficients $\{s_k\}$ are real, this multiresolution analysis always has a scaling function which is supported in the interval $[0, \infty)$; it also has one supported in $(-\infty, 0]$. The proof of the Theorem and the details to these remarks can be found in Subsection 2.1.

1.2

The conclusion of Theorem 1 may fail if $\{s_k\}$ is not assumed to be finite. For example, there are sequences $\{s_k\}$ which satisfy both (3) and (4) and decay faster than the reciprocal of any polynomial such that the corresponding function φ given by (5) fails to generate a multiresolution analysis.

Nevertheless this theorem can be extended to sequences which are not finite and/or do not necessarily satisfy (3). For example the arguments in Subsection 2.1 show that the conclusion of Theorem 1 holds whenever the bi-infinite sequence $\{s_k\}$ satisfies (4) and enjoys exponential decay as $|k| \to \infty$.

1.3

If $\{s_k\}$ is a scaling sequence, consider the corresponding periodic function $S(\xi)$ defined by (6). Recall that the constraints (3) and (4) on $\{s_k\}$ are equivalent to the conditions

$$|S(\xi)|^2 + |S(\xi+\pi)|^2 = 1 \tag{8}$$

and

$$S(0) = 1 \tag{9}$$

on $S(\xi)$. Furthermore, relation (1) becomes

$$\hat{\varphi}(\xi) = S(\xi/2)\hat{\varphi}(\xi/2), \tag{10}$$

and relation (2) is equivalent to

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi + 2\pi k)|^2 = 1\,. \tag{11}$$

The interplay between $\{s_k\}$, $S(\xi)$, and the scaling equation (1) may be made more succinct if to $\{s_k\}$ one associates the train of Dirac delta distributions

$$s(x) = \sum_k s_k \delta(x-k). \tag{12}$$

Then

$$S(\xi) = \frac{\hat{s}(\xi)}{2} \tag{13}$$

and equations (1) and (10) become

$$\varphi(x) = s * \varphi(2x) \tag{14}$$

and

$$\hat{\varphi}(\xi) = \frac{\hat{s}(\xi/2)}{2}\hat{\varphi}(\xi/2) \tag{15}$$

respectively.

Given a scaling sequence $\{s_k\}$ which satisfies the constraints (3) and (4) and the corresponding periodic function $S(\xi)$ defined by (6) consider the function $S_N(\xi)$ defined by

$$S_N(\xi) = S(N\xi), \tag{16}$$

where N is a positive integer. Clearly, S_N is also a 2π periodic function and if N is odd it also satisfies (8) and (9). In other words

$$|S_N(\xi)|^2 + |S_N(\xi+\pi)|^2 = 1 \tag{17}$$

and

$$S_N(0) = 1 \tag{18}$$

whenever N is an odd integer.

If $s(x)$ is the train of delta functions which corresponds to $\{s_k\}$ and $S(\xi)$ then the train $s_N(x)$ corresponding to $S_N(\xi)$ is given by

$$s_N(x) = \frac{1}{N}s(x/N), \tag{19}$$

which, in view of the fact that $\delta(x/N) = N\delta(x)$, may be expressed as

$$s_N(x) = \sum_k s_k\delta(x - Nk). \tag{20}$$

The corresponding scaling sequence $\{s_{N,k}\}_k$ may be somewhat awkwardly described by

$$s_{N,k} = \begin{cases} s_m & \text{if } k = mN, m = 0, \pm 1, \pm 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

In other words

$$s_N(x) = \sum_k s_{N,k}\delta(x - k). \tag{22}$$

In view of the relationship (21) between $\{s_k\}$ and $\{s_{N,k}\}_k$, we say that $\{s_{N,k}\}_k$ is a dilate of $\{s_k\}$. Similarly, we say that S_N is a dilate of S.

If $\varphi(x)$ is the canonical scaling function given by (5) associated with $\{s_k\}$ then the function φ_N defined by

$$\varphi_N(x) = \frac{1}{N}\varphi(\frac{x}{N}) \tag{23}$$

is the corresponding scaling function associated with the sequence $\{s_{N,k}\}$.

If N is an odd integer, in view of (17) and (18), the sequence $\{s_{N,k}\}_k$ is also a scaling sequence which satisfies (3) and (4). Unfortunately if $N > 1$, the corresponding scaling function φ_N necessarily fails to satisfy (2), namely, the collection $\{\varphi_N(x-k)\}_{k\in\mathbb{Z}}$ fails to be orthogonal. See Section 2.3 for more details.

The above observations give rise to an easy method for constructing scaling sequences which satisfy (3) and (4) but whose corresponding scaling functions given by (5) fail to satisfy (2).

1.4

In the previous section we described a method for constructing scaling sequences which

- satisfy (3) and (4)
- fail to give rise via (5) to scaling functions φ which satisfies (2).

It is not immediately clear whether there are scaling sequences which enjoy these properties and which do not arise in this way. In view of this it is natural to ask whether every sequence which enjoys the properties listed above is an appropriate dilate $\{s_{N,k}\}$, with N odd and > 1, of another scaling sequence $\{s_k\}$.

This question was answered in the negative by Cohen and Sun [3]. Because their construction also resolves a more delicate question, see Subsection 1.5, it is unnecessarily complicated. Here we present an alternate, more elementary solution.

Observe that the existence of a finite scaling sequence $\{s_k\}$ which has the appropriate properties and is not a dilate of another scaling sequence follows from the existence of a trigonometric polynomial

$$S(\xi) = \sum_{k=0}^{n-1} s_k e^{-ik\xi} \tag{24}$$

which satisfies (8) and (9), gives rise via (5) to a scaling function φ which fails to satisfy (2), and has the property that $S(\xi/N)$ is not 2π periodic for any integer $N > 1$.

Proposition 1 *For every even integer n which is ≥ 8 there are trigonometric polynomials S of the form (24) which enjoy the following properties:*

(i) The coefficients $\{s_k\}$ are real, $s_0 \neq 0$, and $s_{n-1} \neq 0$.

(ii) $S(\xi)$ satisfies (8) and (9).

(iii) The function φ defined via (5) fails to satisfy (2).

(iv) $S(\xi/N)$ is not 2π periodic for any integer $N > 1$.

Our construction of the polynomials S whose existence is guaranteed by the above proposition is based on the following lemma

Lemma 1 *If S is a 2π periodic trigonometric polynomial and has $1 + e^{-iN\xi}$ as a factor, where N is an odd integer greater than 1, then the function φ defined by (5) fails to satisfy (2).*

The idea is to construct a 2π periodic trigonometric polynomial S which satisfies (8), (9), and the conditions of Lemma 1 but is not a dilate of another 2π periodic trigonometric polynomial. In the case $N = 3$, take

$$S(\xi) = \frac{1 + e^{-i3\xi}}{2} F(\xi), \tag{25}$$

where $F(\xi)$ is a trigonometric polynomial chosen so that S satisfies (8) and (9). Of course the choice $F(\xi) = 1$ does the job but the resulting $S(\xi)$ is a dilate of $\frac{1+e^{-i\xi}}{2}$.

Lemma 2 *For every even integer n which is ≥ 4 there are trigonometric polynomials $F(\xi)$ of the form*

$$F(\xi) = \sum_{k=0}^{n} c_k e^{-ik\xi}, \tag{26}$$

such that

- *$c_0 \neq 0$ and $c_n \neq 0$*
- *The polynomial S defined by (25) satisfies (8) and (9)*
- *This polynomial can be chosen so that $c_1 \neq 0$.*

The last item of the Lemma 2 guarantees that $S(\xi/N)$ is not 2π periodic for any integer $N > 1$.

The coefficients $\{c_k\}$ in (26) are solutions to a system of quadratic equations. A specific numerical solution to this system in the case $n = 4$ is given by

$$\{c_0, c_1, c_2, c_3, c_4\} = \{0.280046, -0.553373, 0.609149, 0.441001, 0.223178\}.$$

This gives rise to a polynomial $S(\xi)$ in the form (24) with $n = 8$ and which satisfies the conclusion of Proposition 1. The coefficients $\{s_0, ..., s_7\}$ are

$$\{0.140023, -0.276687, 0.304574, 0.360523, \\ -0.165098, 0.304574, 0.220500, 0.111589\}.$$

Details are in Section 2.4 .

1.5

In view of Lemma 1 one may ask whether the presence of the factor $1 + e^{-iN\xi}$, N odd and greater than 1, is necessary in the polynomials $S(\xi)$ whose existence is guaranteed by Proposition 1. The example constructed by Cohen and Sun [3] shows that the answer is no.

Here, by slightly modifying the example given in [3], we construct the polynomial with real coefficients of minimal degree which does not have the factor $1 + e^{-iN\xi}$, N odd and greater than 1, and enjoys properties *(i)-(iv)* of Proposition 1.

Consider the 2π periodic trigonometric polynomial

$$G(\xi) = \frac{e^{i9\xi} + 1}{2}$$

whose roots in $(-\pi, \pi]$ are

$$\{\frac{-7\pi}{9}, \frac{-5\pi}{9}, \frac{-3\pi}{9}, \frac{-\pi}{9}, \frac{\pi}{9}, \frac{3\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi\}.$$

This polynomial has factors

$$G_1(\xi) = \frac{(e^{i\xi} - e^{-i\pi/3})(e^{i\xi} - e^{i\pi/3})}{2}$$

and

$$G_2(\xi) = \frac{G(\xi)}{G_1(\xi)}.$$

Observe that the set of roots of $G_2(\xi)$ in $(-\pi, \pi]$ is the set

$$\Omega = \{\frac{-7\pi}{9}, \frac{-5\pi}{9}, \frac{-\pi}{9}, \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi\}.$$

Define the polynomial $H(\xi)$ via

$$H(\xi) = |G_2(\xi)|^2\{|G_1(\xi)|^2 + c|G_2(\xi + \pi)|^2 \cos\xi\} \tag{27}$$

where the real constant c is chosen so that the expression in braces is positive. This polynomial enjoys the following properties:

- $H(\xi)$ is a non-negative 2π periodic trigonometric polynomial whose set of roots in $(-\pi, \pi]$ is the set Ω.
- $H(-\xi) = H(\xi)$.
- $H(\xi) + H(\xi + \pi) = 1$.

The last two items are can be easily seen by first re-expressing $H(\xi)$ as

$$H(\xi) = |G(\xi)|^2 + c|G_2(\xi)|^2|G_2(\xi+\pi)|^2 \cos\xi$$

and observing the behavior of each of the two summands. These properties imply that

$$H(\xi) = \sum_{k=0}^{15} a_k \cos k\xi$$

which, in view of the lemma of Riesz, can be factored as

$$H(\xi) = |S(\xi)|^2 \tag{28}$$

where

$$S(\xi) = \sum_{k=0}^{15} s_k e^{-ik\xi} \tag{29}$$

with real coefficients s_k.

Proposition 2 *The 2π periodic polynomial $S(\xi)$ defined by (28) and (29) satisfies properties (i)-(iv) of Proposition 1 and does not have $1+e^{-iN\xi}$ as a factor, for any integer $N > 1$.*

1.6

Up to this point we have been mainly concerned with the canonical solution of (1), that is, the solution φ described by (5). Since there are other solutions of (1), which are not constant multiples of the canonical solution, it is natural to ask when such solutions satisfy (2). The following theorem should resolve most of these questions.

Theorem 2 *Suppose $\{s_k\}$ is a finite scaling sequence which satisfies (3) and (4) and φ is the function whose formula is given by (5).*

- *If φ fails to satisfy (2) then every solution of (1) also fails to satisfy (2).*
- *If φ satisfies (2) then λ is another solution of (1) which satisfies (2) if and only if*

  $$\hat{\lambda}(\xi) = h(\xi)\hat{\varphi}(\xi)$$

  where $h(2\xi) = h(\xi)$ and $|h(\xi)| = 1$ almost everywhere on $I\!R$.

This result may be roughly rephrased as follows: *If λ is a solution of (1) which satisfies (2) then the canonical solution φ also satisfies (2) and*

$$|\hat{\lambda}(\xi)| = |\hat{\varphi}(\xi)|$$

almost everywhere on $I\!R$.

1.7

Finally we mention that the examples in Subsections 1.4 and 1.5 have lowest degree possible. More precisely,

Proposition 3 *Every trigonometric polynomial S of the form (24) which satisfies the conditions of Proposition 1 has $n \geq 8$. If, in addition, S is required to not have $1 + e^{-iN\xi}$, N odd and greater than 1, as a factor then $n \geq 16$.*

2 Details

These sections contain details omitted in the previous sections. We use the convention, employed in [1], that the details to Section 1.n are contained in Section 2.n .

2.1

We begin by recalling the following facts from [7] :

Proposition 4 *A function φ in $L^2(I\!R)$ generates a dyadic multiresolution analysis if and only if φ enjoys the following properties :*

- *There is a 2π periodic measurable function $S(\xi)$ such that*

$$\hat{\varphi}(\xi) = S(\xi/2)\hat{\varphi}(\xi/2)\,. \tag{30}$$

- *For almost all ξ*

$$\sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2\pi k)|^2 > 0\,. \tag{31}$$

- *For every finite interval I*

$$\lim_{j\to\infty} \frac{1}{|2^{-j}I|} \int_{2^{-j}I} \left\{ |\hat{\varphi}(\xi)|^2 \Big/ \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2\pi k)|^2 \right\} d\xi = 1, \tag{32}$$

  where $2^{-j}I = \{x : 2^j x \in I\}$.

Proposition 5 *Suppose φ is a function in $L^2(I\!R)$ which enjoys the properties listed in Proposition 4 and $P(\xi)$ is a measurable 2π periodic function such that*

$$|P(\xi)|^2 = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2\pi k)|^2.$$

Then the function φ_0 defined by

$$\hat{\varphi}_0(\xi) = \frac{\hat{\varphi}(\xi)}{P(\xi)} \tag{33}$$

is a scaling function for the multiresolution analysis generated by φ.

Suppose $\{s_k\}$ and φ are as in the hypothesis of Theorem 1. Then φ is in $L^2(\mathbb{R})$, has compact support and satisfies (30) with S given by (6); see [4].

Define the periodic function Φ via

$$\Phi(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2\pi k)|^2. \tag{34}$$

By virtue of the Poisson summation formula, Φ is a trigonometric polynomial and hence $\Phi(\xi) > 0$ almost everywhere.

Since S satisfies (8) and (9), it follows that $S(\pi) = 0$ and hence, in view of (5), $\varphi(2\pi k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Thus $|\hat{\varphi}(\xi)|^2/\Phi(\xi)$ is continuous in a neighborhood of zero and equals one for $\xi = 0$. This implies that (32) holds and, together with the previous observations, allows us to conclude that all the conditions in Proposition 4 are satisfied, so the proof of Theorem 1 is complete.

Observe that

$$\hat{\varphi}_0(\xi) = \frac{\hat{\varphi}(\xi)}{P(\xi)} = \frac{S(\xi/2)\hat{\varphi}(\xi/2)}{P(\xi)} = \frac{S(\xi/2)P(\xi/2)\hat{\varphi}_0(\xi/2)}{P(\xi)}$$

so that φ_0 satisfies the scaling equation

$$\hat{\varphi}_0(\xi) = S_0(\xi/2)\hat{\varphi}_0(\xi/2), \tag{35}$$

where

$$S_0(\xi) = \frac{S(\xi)P(\xi)}{P(2\xi)}. \tag{36}$$

Note that $S_0(\xi)$ satisfies (8) and (9) as it should. Also note that if φ satisfies (2) then $\Phi(\xi) \equiv 1$ so that P can also be taken to be $\equiv 1$ and, in this case, $\varphi_0 = \varphi$.

For the function P in (33) one can simply take $P(\xi) = \sqrt{\Phi(\xi)}$, the positive square root of $\Phi(\xi)$. If $\Phi(\xi) = \Phi(-\xi)$, for example this is the case when all the coefficients $\{s_k\}$ are real, then by virtue of the Riesz lemma, see [4], $P(\xi)$ can be chosen to be a trigonometric polynomial; in this case $\sigma(e^{-i\xi}) = S_0(\xi)$ is a rational function of $z = e^{-i\xi}$.

Let $P(\xi) = p(e^{-i\xi})$ be a 2π periodic trigonometric polynomial satisfying

$$|P(\xi)|^2 = \Phi(\xi) \tag{37}$$

and observe that the following are also 2π periodic trigonometric polynomials satisfying (37).

- $e^{im\xi}P(\xi)$ for any integer m and
- $P(\xi)(1 - \bar{a}e^{-i\xi})/(e^{-i\xi} - a)$ where a is a root of $p(z)$.

The last item implies that $P(\xi) = p(e^{-i\xi})$ can be chosen so that all the roots a of $p(z)$ satisfy either $|a| \leq 1$ or $|a| \geq 1$.

If P is chosen so that all the roots of $p(z)$ are ≥ 1 then, in view of the fact that $|S_0(\xi)| \leq 1$, all the poles b of the rational function $\sigma(z)$, if any, satisfy $|b| > 1$ and, by multiplying $P(\xi)$ by additional factors of $e^{-i\xi}$ if necessary, we see that $S_0(\xi)$ can be expressed as

$$S_0(\xi) = \sum_{k=0}^{\infty} r_k e^{-ik\xi}$$

where $\limsup_{k\to\infty} |r_k|^{1/k} < 1$. In this case, since $\varphi_0(x)$ is the solution of the two scale difference equation

$$\varphi_0(x) = \sum_{k=0}^{\infty} r_k \varphi_0(2x - k)$$

via fixed-point iteration starting with the indicator function of the interval $[0, 1]$, it follows that $\varphi_0(x)$ has support in $[0, \infty)$.

An analogous computation produces $\varphi_0(x)$ with support in $(-\infty, 0]$.

2.2

To see that the conclusion of Theorem 1 may fail if $\{s_k\}$ does not decay sufficiently rapidly, consider the 2π periodic function $S(\xi)$ which is defined as follows: begin with any non-negative C^∞ function, $h(\xi)$, which is supported in $I_\epsilon = \{\xi : |\xi| \leq \pi + \epsilon\}$ and is positive on $I_{\epsilon/2}$; here ϵ satisfies $0 < \epsilon < \pi$. Define $g(\xi)$ by

$$g(\xi) = \frac{h(\xi)}{\sqrt{\sum_{k\in\mathbf{Z}} |h(\xi + 2\pi k)|^2}}$$

and note that

$$\sum_{k\in\mathbf{Z}} |g(\xi + 2\pi k)|^2 = 1\,. \tag{38}$$

Let

$$T(\xi) = \sum_{k\in\mathbf{Z}} g(\xi + 4\pi k)$$

and finally define $S(\xi)$ via

$$S(\xi) = T(2N\xi)$$

where N is a positive odd integer. Since $S(\xi)$ is an infinitely differentiable 2π periodic function, we may write

$$S(\xi) = \frac{1}{2} \sum_{k\in\mathbf{Z}} s_k e^{-ik\xi} \tag{39}$$

where the sequence $\{s_k\}$ decays faster than the reciprocal of any polynomial. In other words, $\{s_k\}$ enjoys the estimates

$$|s_k| \leq C_p(1+|k|)^{-p}$$

for all positve p where the constant C_p is independent of k. In view of (38) and the fact that

$$|T(\xi)|^2 = \sum_{k\in\mathbb{Z}} |g(\xi + 4\pi k)|^2,$$

the function $S(\xi)$ satisfies (8) and (9) which implies that $\{s_k\}$ satisfies (3) and (4).

Now, if φ is the function defined in terms of $S(\xi)$ via (5) then $\hat{\varphi}$ vanishes in a neighborhood of $2k\pi/N$, for all k in $\mathbb{Z}\setminus\{0\}$. Thus if N is greater than 1 then, recalling that N is odd, it follows that

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2\pi k)|^2$$

vanishes in a neighborhood of $2\pi/N$. In view of the second item in Proposition 4, we may conclude that φ fails to generate a multiresolution analysis.

2.3

To see that $\varphi_N(x)$ defined by (23), with an odd integer $N > 1$, necessarily fails to satisfy (2), assume that $\{s_k\}$ is finite and observe that $\varphi(2\pi m) = 0$ for all m in $\mathbb{Z}\setminus\{0\}$. Thus $\hat{\varphi}_N(\xi) = \hat{\varphi}(N\xi)$ vanishes whenever $\xi = 2m\pi/N$ for all m in $\mathbb{Z}\setminus\{0\}$. Hence

$$\Phi_N(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\varphi_N}(\xi - 2\pi k)|^2$$

is a continuous periodic function which vanishes when $\xi = (2\pi/N) + 2\pi m$, $m \in \mathbb{Z}$ and the desired result follows since the orthogonality of the system $\{\varphi_N(x-k)\}_k$ is equivalent to $\Phi_N(\xi)$ =constant.

The general case follows from a similar argument.

2.4

To see Lemma 1 write

$$S(\xi) = \frac{1+e^{-iN\xi}}{2} F(\xi) \tag{40}$$

where $F(0) = 1$ and observe that

$$\hat{\varphi}(\xi) = \frac{1-e^{-iN\xi}}{2iN\xi} \mathcal{F}(\xi) \tag{41}$$

where

$$\mathcal{F}(\xi) = \prod_{k=1}^{\infty} F(\xi/2^k)$$

is an analytic function. From (41) we may conclude that $\hat{\varphi}(\xi)$ vanishes whenever $\xi = 2m\pi/N$ for all m in $\mathbb{Z} \setminus \{0\}$ and the desired result follows from the fact that

$$\Phi(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2\pi k)|^2$$

vanishes when $\xi = (2\pi/N) + 2\pi m$, $m \in \mathbb{Z}$.

To find a suitable polynomial F for Lemma 2 substitute (25) into (8) to get

$$|G(\xi)|^2|F(\xi)|^2 + |G(\xi+\pi)|^2|F(\xi+\pi)|^2 = 1, \tag{42}$$

where $|G(\xi)|^2 = (\cos(\frac{3\xi}{2}))^2$ and, of course, $|G(\xi+\pi)|^2 = (\sin(\frac{3\xi}{2}))^2$. Recall that

$$|G(\xi)|^2 + |G(\xi+\pi)|^2 = 1$$

and observe that this together with (42) implies

$$|G(\xi)|^2\{|F(\xi)|^2 - 1\} + |G(\xi+\pi)|^2\{|F(\xi+\pi)|^2 - 1\} = 0. \tag{43}$$

Now (43) is not difficult to solve for $|F(\xi)|^2$ if we choose it to be of the form

$$|F(\xi)|^2 = |G(\xi+\pi)|^2 Q(\xi+\pi) + 1, \tag{44}$$

where Q is a trigonometric polynomial. Thus

$$|F(\xi+\pi)|^2 = |G(\xi)|^2 Q(\xi) + 1$$

and substituting the last two expressions in (43) results in

$$|G(\xi)|^2|G(\xi+\pi)|^2\{Q(\xi+\pi) + Q(\xi)\} = 0$$

or, more simply,

$$Q(\xi+\pi) + Q(\xi) = 0. \tag{45}$$

It is clear that any linear combination of $\cos((2k-1)\xi)$, $k = 0, 1, 2, \ldots$, solves (45). Namely every trigonometric polynomial Q of the form

$$Q(\xi) = \sum_{k=1}^{l} a_k \cos((2k-1)\xi) \tag{46}$$

is a solution of (45). In particular choosing

$$Q(\xi) = 4a\cos(\xi) \tag{47}$$

results in

(48) $$|F(\xi)|^2 = a(\cos 4\xi + \cos 2\xi - 2\cos\xi) + 1.$$

Details of this calculation can be found at the end of this section.

In general it is clear that if the absolute values of the coefficients a_k in representation (46) are sufficiently small then the right hand side of (44) is positive. In this case the Riesz lemma mentioned above implies the existence of a trigonometric polynomial

(49) $$F(\xi) = \sum_{k=0}^{2l+2} c_k e^{-ik\xi}$$

which satisfies (44). In particular almost all choices of sufficiently small a_k in representation (46) will reult in non-zero c_0, c_1, and c_{2l+2} in (49). In such cases the polynomial $S(\xi)$ described by (25) satisfies both (3) and (4) and $S(\xi/N)$ fails to be 2π periodic for every integer $N \neq 1$.

Returning to (48) note that the maximum of $\cos 4\xi + \cos 2\xi - 2\cos\xi$ occurs at $\xi = \pi$ and is 4. Numerical methods show that the minimum occurs at $\xi = 0.818919$ and is -2.42404; these estimates are accurate to the listed number of decimal places. Hence the right hand side of (48) is positive if

(50) $$-0.25 < a < 0.412534.$$

For such a, the Riesz lemma [4, page 172] implies the existence of the desired trigonometric polynomial

(51) $$F(\xi) = \sum_{k=0}^{4} c_k e^{-ik\xi}$$

which satisfies (48). If $a \neq 0$, then it is clear that both coeficients c_0 and c_4 in (51) are not equal to 0 and the polynomial $S(\xi)$ described by (24) has the property that $S(\xi/N)$ fails to be 2π periodic for every integer $N \neq 1$.

To get an explicit collection of the coefficients c_k for the polynomial $F(\xi)$ given by (51) observe that such coefficients must satisfy the system of equations

$$\begin{aligned}
&c_0^2 + c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1\\
&c_0c_1 + c_1c_2 + c_2c_3 + c_3c_4 = -a\\
&c_0c_2 + c_1c_3 + c_2c_4 = a/2\\
&c_0c_3 + c_1c_4 = 0\\
&c_0c_4 = a/2
\end{aligned}$$

where $a \neq 0$ and satisfies the constraints (50). This system of equations can be easily solved by numerical method such as Newton's method. In the case $a = 0.125$ one such solution is

$$\{c_0, c_1, c_2, c_3, c_4\} = \{0.280046, -0.553373, 0.609149, 0.441001, 0.223178\}.$$

To see (48) recall that

$$|G(\xi)|^2 = (\cos(\frac{3\xi}{2}))^2 = \frac{1+\cos 3\xi}{2}$$

so that if Q is given by (47)

$$|G(\xi)|^2 Q(\xi) = 2a(\cos\xi + \cos 3\xi \cos\xi).$$

Now using

$$\cos 3\xi \cos\xi = \frac{1}{2}(\cos 4\xi + \cos 2\xi)$$

we get

$$|G(\xi)|^2 Q(\xi) = a(\cos 4\xi + \cos 2\xi + 2\cos\xi)$$

and

$$|G(\xi+\pi)|^2 Q(\xi+\pi) = a(\cos 4\xi + \cos 2\xi - 2\cos\xi).$$

Substituting the last formula into (44) gives the desired result.

2.5

To see that the constant c can be chosen so that the expression in braces in formula (27) is positive, note that $|G_1(\xi)|^2$ is positive on $(-\pi,\pi]$ except at $\xi = \pi/3$ and at $\xi = -\pi/3$. Since the set of roots of $|G_2(\xi+\pi)|^2$ in $(-\pi,\pi]$ is the set

$$\Omega + \pi = \{\frac{-8\pi}{9}, \frac{-4\pi}{9}, \frac{-2\pi}{9}, 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}\}$$

there is a positive ϵ such that $|G_2(\xi+\pi)|^2$ is positive for ξ in $J_\epsilon = (\pi/3 - \epsilon, \pi/3+\epsilon) \cup (5\pi/3 - \epsilon, 5\pi/3+\epsilon)$. Now, if

$$m = \inf_{\xi\in[0,2\pi]\setminus J_\epsilon} |G_1(\xi)|^2(\xi)$$

and

$$M = \sup_{\xi\in[0,2\pi]\setminus J_\epsilon} |G_2(\xi+\pi)|^2 .$$

then any c such that $0 < c < m/M$ will do the job.

To see Proposition 2 note that it follows from the properties of $H(\xi)$ that $S(\xi)$ satisfies the first, second and fourth conditions of Proposition 1 . Furthermore, since Ω is the set of roots of $S(\xi)$ in $(-\pi,\pi]$, $S(\xi)$ does not have $1+e^{-iN\xi}$ as a factor. However, the third condition of Proposition 1 needs some explanation.

Observe that the set $\Omega+\pi$ listed above has the following properties:

- $\Omega+\pi$ is a non-trivial invariant cycle in $(-\pi,\pi]$ for the operation $\xi \mapsto 2\xi \mod 2\pi$.

- $|S(\xi)| = 1$ for all ξ in $\Omega + \pi$.

In view of these properties Cohen's result concerning such cycles implies that $S(\xi)$ satisfies the third condition of Proposition 1. See [4, page 188] or [2, 3, 5] for more details.

Before closing this subsection we mention that the construction of $S(\xi)$ here is really analogous to that in Subsection 2.4 with $|G_1(\xi)|^2$ playing the role here that 1 played there.

2.6

To see Theorem 2, suppose $\{s_k\}$ is a finite scaling sequence which satisfies (3) and (4), φ is the function whose formula is given by (5), and λ is another solution to (1). Then

$$\hat{\lambda}(\xi) = h(\xi)\hat{\varphi}(\xi) \tag{52}$$

where h is a function which satisfies

$$h(2\xi) = h(\xi) \tag{53}$$

almost everywhere on $I\!R$. Observe, that if $\hat{\lambda}$ is essentially bounded then so is h, namely,

$$\text{ess } \sup_{\xi \in R} |h(\xi)| \leq \text{ess } \sup_{\xi \in R} |\hat{\lambda}(\xi)| \,. \tag{54}$$

Inequality (54) follows from

$$\text{ess } \sup_{\xi \in R} |\hat{\lambda}(\xi)| \geq \text{ess } \sup_{|\xi|<\delta} |h(\xi)| \inf_{|\xi|<\delta} |\hat{\varphi}(\xi)| \,,$$

the fact that $\inf_{|\xi|<\delta} |\hat{\varphi}(\xi)|$ is arbitrarily close to 1 for sufficiently small δ, and because of (53), for any positive δ

$$\text{ess } \sup_{|\xi|<\delta} |h(\xi)| = \text{ess } \sup_{\xi \in R} |h(\xi)| \,.$$

Assume φ fails to satisfy (2). To see that this implies that λ also fails to satisfy (2) suppose, on the contrary, that it does satisfy (2). Then

$$\sum_{k=-\infty}^{\infty} |\hat{\lambda}(\xi - 2\pi k)|^2 = 1 \tag{55}$$

and, by virtue of (54),

$$\text{ess } \sup_{\xi \in R} |h(\xi)| \leq 1 \,. \tag{56}$$

Thus, we may write

$$\sum_{k=-\infty}^{\infty} |\hat{\lambda}(\xi - 2\pi k)|^2 \leq \Phi(\xi) \tag{57}$$

where Φ is the trigonometric polynomial

$$\Phi(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\xi - 2\pi k)|^2 .$$

Since φ fails to satisfy (2) the polynomial $\Phi(\xi)$ is arbitrarily small on a set of positive measure which, in view of (57), contradicts (55).

To see the second assertion of the theorem assume that λ satisfies (2). Then by virtue of (55) and (56), we may write

$$|h(\xi)\hat{\varphi}(\xi)|^2 + \sum_{k\in \mathbf{Z}\setminus\{0\}} |\hat{\varphi}(\xi - 2\pi k)|^2 \geq 1 . \tag{58}$$

Now, since $\Phi(\xi) \leq 1$ and $\hat{\varphi}(0) = 1$, we may conclude that for any ϵ satisfying $0 < \epsilon < 1/2$

$$|h(\xi)|^2 \geq 1 - \epsilon \tag{59}$$

whenever ξ is in a sufficiently small neighborhood of 0. Since

$$\text{ess} \inf_{|\xi|<\delta} |h(\xi)| = \text{ess} \inf_{\xi\in R} |h(\xi)| .$$

for any positive δ, (59) allows us to conclude that

$$\text{ess} \inf_{\xi\in R} |h(\xi)| \geq 1 .$$

The last inequality together with (56) imply that

$$|h(\xi)| = 1 \tag{60}$$

almost everywhere. Conversely, if h satisfies (60) and $\Phi(\xi) = 1$ then (55) follows from (52).

2.7

Proposition 3 is an easy but tedious consequence of the characterization of zero sets of those polynomials $S(\xi)$ which fail to generate a φ which satisfies (2). These sets are characterized in [3, 5]. The details will appear elsewhere.

3 Remarks and acknowledgements

Section 1.1 was motivated by Lawton's result [6] which asserts that if $\{s_k\}$ is a finite sequence satisfying (3) and (4) and if φ is the function defined by (5) then the collection of functions $\{2^{k/2}\psi(2^k x - j)\}_{k,j\in \mathbf{Z}}$, where

$$\psi(x) = \sum (-1)^k \bar{s}_{1-k}\varphi(2x - k) \tag{61}$$

is always a tight frame. See also [4, page 178]. The material in Section 1.1 implies that there are also orthonormal wavelet bases associated with $\{s_k\}$ which can be derived from the scaling function φ_0 in the usual way.

The questions addressed in this paper arose in lectures WRM presented during the 1993 spring semester at the University of Connecticut. Apparently the question raised in Subsection 1.4 was first publicly raised by D. Pollen, see [3].

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References

[1] A. P. Calderon, Intermediate spaces and interpolation, the complex method, *Studia Math.*, **24**, (1964), 113-190.

[2] A. Cohen, Ondelettes, analyses multirésolutions, et traitement numérique du signal, Ph.D. Thesis, Université Paris, Dauphine, 1990.

[3] A. Cohen and Q. Sun, An arithmetic characterization of the conjugate quadratic filters associated to orthonormal wavelet bases, *SIAM J. Math. Anal.*, **24**, (1993), 1355-1360.

[4] I. Daubechies, *Ten Lectures on Wavelets*, CBMS, **61**, SIAM, Philadelphia, 1992.

[5] K. Gröchenig, Orthogonality criteria for compactly supported scaling functions, preprint.

[6] W. Lawton, Tight frames of compactly supported wavelets, *J. Math. Phys.*, **31**, (1990), 1898-1901.

[7] W. R. Madych, Some elementary properties of multiresolution analyses of $L^2(\mathbb{R}^n)$, in *Wavelets : A Tutorial in Theory and Applications*, C. K. Chui (ed.), Academic Press, San-Diego, 1992, 259-294.